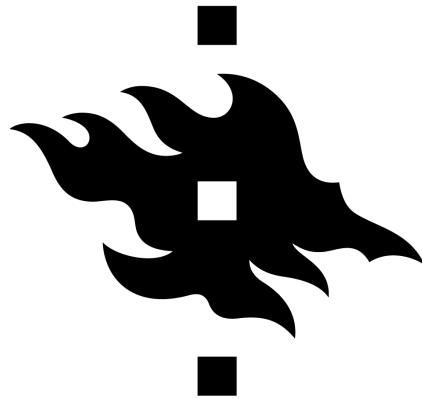


# Brouwer's Theorem on the Invariance of Domain

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<p>The purpose of this thesis is to present some dimension theory of separable metric spaces, and with the theory developed, prove Brouwer's Theorem on the Invariance of Domain. This theorem states, that if we embed a subset of the <math>n</math>-dimensional Euclidean space into the aforementioned space, this embedding is an open map.</p> <p>We begin by revising some elementary theory of point-set topology, that should be familiar to any graduate student in mathematics. Drawing from these rudiments, we move on to the concept of dimension.</p> <p>The dimension theory presented is based on the notion of the small inductive dimension. We define this dimension function for regular spaces and state and prove various results that hold for this function. Although this dimension function is defined on regular spaces, we mainly focus on separable metric spaces. Among other things, we prove that the small inductive dimension of the Euclidean <math>n</math>-space is exactly <math>n</math>. This proof makes use of the famous Brouwer Fixed-Point Theorem, which we naturally also prove. We give a combinatorial proof of the Fixed-Point Theorem, which relies on Sperner's lemma.</p> <p>We move on to develop some theory regarding the extensions of functions. These various results on extensions allow us to finally prove the theorem that lent its name to this thesis: Brouwer's Theorem on the Invariance of Domain.</p>			
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## Introduction

The aim of this thesis is to present rudimentary dimension theory, in particular dimension theory regarding separable metric spaces, and with the tools developed, prove Brouwer's Theorem on the Invariance of Domain. This theorem states, that if  $A$  is a subset of the Euclidean space  $\mathbb{R}^n$ , an embedding  $h: A \rightarrow \mathbb{R}^n$  is an open map. This result is simple in the way, that anyone familiar with elementary topology can understand the meaning of it, and yet as we shall see, the proof is not so simple.

There are several notions of dimension in topology, but our focus in this thesis is only on the small inductive dimension. Should the reader be so inclined, Professor Ryszard Engelking's *Dimension Theory* [1] contains a comprehensive exposition – among small inductive dimension – of other types of dimension, i.e. large inductive dimension and covering dimension.

We begin by a revision of topological concepts that should be familiar to any graduate student of mathematics. This chapter relies mostly on the two books by Professor Jussi Väisälä, [6] and [7]. The book *Topology* by Professor James Munkres [5] has served not so much as a mathematical reference, rather as a stylistic guide on language. The definition of a separator between two subsets of some space is from [1]. The proof of Urysohn's lemma is from Engelking's *General Topology* [2].

Chapter 3 introduces the concept of small inductive dimension, or simply dimension as we refer to it in this thesis. We begin by defining the notion of 0-dimensionality before moving on to the general notion of dimension. We cover many basic results of the dimension theory of separable metric spaces, although some results hold for more general spaces, namely regular spaces. The main source for this chapter is [4]. The book by Engelking [1] has also contributed as a source for this chapter.

In Chapter 4 we develop some theory of simplexes in order to prove the

famous Brouwer Fixed-Point Theorem. With the use of this theorem, we are able to show that the dimension of the Euclidean space  $\mathbb{R}^n$  is exactly  $n$ , as one should expect. The section on simplexes is almost entirely based on [2]. The lecture notes of a combinatorics course taught at Princeton University by Jacob Fox [3] provided help in the proof of Sperner's lemma. From Proposition 4.18 to the end of Chapter 4 we rely on [4].

In Chapter 5 we develop some theory concerning extensions of functions and use this theory to prove the Brouwer's Theorem on the Invariance of Domain, which lent its name to this thesis. This chapter is based on [4], with the exception of the Tietze Extension Theorem, the proof of which is from [2].

I would like to express my gratitude to my thesis supervisors, Drs. Erik Elfving and Pekka Pankka, for all the help and guidance they have so kindly offered to me.



## Topological spaces

**Definition 2.1** (Topology). Let  $X$  be some set and  $\mathcal{T}$  a collection of subsets of  $X$ , i.e.  $\mathcal{T} \subset \mathcal{P}(X)$ , where  $\mathcal{P}(X)$  is the power set of  $X$ . The collection  $\mathcal{T}$  is called a *topology* on  $X$  if the following conditions hold:

- (T1) Any union of sets in  $\mathcal{T}$  is an element of  $\mathcal{T}$ .
- (T2) Any finite intersection of sets in  $\mathcal{T}$  is an element of  $\mathcal{T}$ .
- (T3) The empty set  $\emptyset$  and the whole set  $X$  are members of  $\mathcal{T}$ .

An ordered pair  $(X, \mathcal{T})$ , where  $\mathcal{T}$  is some topology on the set  $X$  is called a topological space. The sets in  $\mathcal{T}$  are the so called open sets in  $X$ . A set in  $X$  is said to be closed, if its complement is open in  $X$ . It might seem intuitive to think that a set being open implies that it is not closed, and vice versa. This is not the case. For an easy counterexample, one can see from (T3) of Definition 2.1 that the whole space  $X$  and the empty set  $\emptyset$  are always both open and closed.

**Definition 2.2** (Subspace topology). Let  $(X, \mathcal{T})$  be a topological space. If  $Y$  is a subset of  $X$ , the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on  $Y$  called the *subspace topology*.

**Definition 2.3** (Product topology). Suppose we have some indexed collection of topological spaces  $(X_j, \mathcal{T}_j)$  where  $j \in J$  and  $J$  is some index set. The topology, called the *product topology* on the Cartesian product

$$X := \prod_{j \in J} X_j$$

is the coarsest topology for which the projection maps  $\text{pr}_j: X \rightarrow X_j$  are continuous. We recall, that if  $U$  is a non-empty open set in  $X$ , then  $\text{pr}_j U = X_j$  except for a finite number of indexes  $j$  (see Väisälä [7][Lause 7.6, p. 49]). If  $X_j = Y$  for all  $j \in J$ , we write  $X = Y^J$ .

**Definition 2.4** (Neighbourhood). Let  $(X, \mathcal{T})$  be a topological space. Let  $x$  be an element of  $X$  and  $A$  a subset of  $X$ . If  $x \in U \in \mathcal{T}$ , then  $U$  is called a *neighbourhood* of  $x$ . The analogous definition holds for sets: if  $A \subset U \in \mathcal{T}$ , then  $U$  is a neighbourhood of  $A$ .

**Proposition 2.5.** *A subset  $A$  of a topological space  $(X, \mathcal{T})$  is open, if and only if every element  $x \in A$  has a neighbourhood  $U_x$ , which is included in  $A$ .*

*Proof.* Suppose  $A$  is open. Now we can choose  $A = U_x$  for every  $x \in A$ . Suppose then that every element  $x \in A$  has a neighbourhood  $U_x$ , which is included in  $A$ . Now we may write  $A = \bigcup_{x \in A} U_x$ , whence it follows that  $A$  is open as a union of open sets.  $\square$

Oftentimes it is difficult to specify the topology on  $X$  by describing the whole collection  $\mathcal{T}$  of open sets. In most cases we can specify a smaller collection of subsets of  $X$ , and define the topology using this smaller collection.

**Definition 2.6** (Basis). Suppose  $\mathcal{T}$  is a topology on a space  $X$ . We call a collection  $\mathcal{B} \subset \mathcal{P}(X)$  a *basis* for the topology on  $X$ , if

- $\mathcal{B} \subset \mathcal{T}$
- Every open set  $U \neq \emptyset$  can be expressed as a union of some sets in  $\mathcal{B}$ .

**Proposition 2.7.** *Suppose  $(X, \mathcal{T})$  is a topological space. The collection  $\mathcal{B} \subset \mathcal{P}(X)$  is a basis for  $\mathcal{T}$ , if and only if*

- (1)  $\mathcal{B} \subset \mathcal{T}$
- (2) If  $x \in U \in \mathcal{T}$ , there exists a  $B \in \mathcal{B}$  such that  $x \in B \subset U$ .

*Proof.* Suppose  $\mathcal{B}$  is a basis for  $\mathcal{T}$ . Then by definition (1) holds. Suppose  $x \in U \in \mathcal{T}$ . Since the set  $U$  can be expressed as a union of some sets in  $\mathcal{B}$ , there exists some basis set  $B \ni x$  in  $\mathcal{B}$  such that  $B \subset U$ .

Suppose (1) and (2) hold. Let  $U$  be a non-empty open set. Now for every  $x \in U$ , we find some set  $B_x \ni x$  from the collection  $\mathcal{B}$ , that is included in  $U$ . Now we have  $U = \bigcup_{x \in U} B_x$ . This completes the proof.  $\square$

Of particular interest in the field of topology are functions that preserve topological properties. These functions are called homeomorphisms. We recall, that a function  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are topological spaces, is said to be continuous, if for every open set  $U \subset Y$ , the preimage  $f^{-1}U$  is open in  $X$ .

**Definition 2.8** (Homeomorphism, embedding). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A continuous bijection  $f: X \rightarrow Y$  is called a *homeomorphism*, if the inverse function  $f^{-1}: Y \rightarrow X$  is also continuous. If a function  $g: X \rightarrow gX$ , where  $gX \subset Y$  is a homeomorphism, the function  $g: X \rightarrow Y$  is called an *embedding*. If  $f$  is a homeomorphism between the spaces  $X$  and  $Y$ , we say that these spaces are homeomorphic and we write  $f: X \approx Y$ .

**Definition 2.9** (Closure). Let  $(X, \mathcal{T})$  be a topological space. The *closure* of the subset  $A \subset X$  is the set

$$\overline{A} := \{x \in X : U \cap A \neq \emptyset, \text{ for every neighbourhood } U \text{ of } x\}.$$

When dealing with subspaces of some topological space  $X$ , we may denote the closure operation as  $\text{cl } A$ , where  $A$  is some subset of  $X$ . Now if  $Y$  is a subspace of the space  $X$ ,  $\text{cl}_X A$  denotes the closure of the set  $A$  in the whole space  $X$ , whereas  $\text{cl}_Y A$  denotes the closure of  $A$  in the subspace  $Y$ .

A point  $x \in X$  is called a *limit point* of  $A$ , if every neighbourhood  $V$  of  $x$  contains some point of  $A$  distinct from  $x$ . If, on the other hand, there exists some neighbourhood  $U$  of  $x$  such that  $U \cap A = \{x\}$ , then  $x$  is called an *isolated point* of  $A$ .

**Definition 2.10** (Interior, exterior, boundary). Let  $X$  be a topological space, and let  $x \in X$  and  $A \subset X$ . The point  $x$  is called an *interior point* of  $A$ , if  $x$  has some neighbourhood  $U \subset A$ . If  $x$  has a neighbourhood  $U \subset \mathbb{C}A$ , then  $x$  is an *exterior point* of  $A$ . If  $x$  is neither an interior nor an exterior point of  $A$ , we say that  $x$  is a *boundary point* of  $A$ . The sets of interior, exterior, and boundary points of  $A$  are denoted as  $\text{int}A$ ,  $\text{ext}A$ , and  $\partial A$  respectively.

**Proposition 2.11.** Suppose  $A$  and  $B$  are subsets of a topological space  $(X, \mathcal{T})$ .

1.  $A \subset \overline{A}$ .
2.  $\overline{A}$  is always closed.
3. If  $B$  is closed in  $X$ , and  $A \subset B$ , then  $\overline{A} \subset B$ .

4.  $\overline{A}$  is the smallest closed subset of  $X$  containing  $A$ .
5. If  $A \subset B$ , then  $\overline{A} \subset \overline{B}$ .
6.  $A$  is closed in  $X$ , if and only if  $A = \overline{A}$ .
7.  $\text{int}A \subset A$ ,  $\text{ext}A \subset \mathbb{C}A$ .
8.  $A$  is open, if and only if  $A = \text{int}A$ .
9.  $\text{ext}A = \mathbb{C}\overline{A}$ ,  $\text{int}A = \mathbb{C}\mathbb{C}\overline{A}$ ,  $\overline{A} = \text{int}A \cup \partial A = A \cup \partial A$ .
10.  $\partial A = \overline{A} \cap \mathbb{C}\overline{A} = \overline{A} \setminus \text{int}A$ , and  $\partial A$  is always closed.
11.  $\partial A = \partial \mathbb{C}A$ .
12. If  $A$  is open, then  $\partial A = \overline{A} \setminus A$ .

*Proof.* See Väisälä [6, Lause 6.8, p. 48 and Lause 8.3, p. 60].  $\square$

**Proposition 2.12.** *A subset  $A$  of a topological space  $X$  is both open and closed, if and only if  $A$  has an empty boundary.*

*Proof.* Suppose, that  $\emptyset \neq A \neq X$ , since the result is evident otherwise.

Suppose  $A \subset X$  is open and closed. From Proposition 2.11 it follows that  $\partial A = \overline{A} \setminus A$ , since  $A$  is open, and  $A = \overline{A}$ , since  $A$  is closed. Hence  $\partial A = \emptyset$ .

Suppose then, that  $\partial A = \emptyset$ . Now from Proposition 2.11 we get that  $\overline{A} = A \cup \partial A = A$ , hence  $A$  is closed. Also  $\partial A = \overline{A} \setminus \text{int}A = \emptyset$ . Hence  $\overline{A} = A = \text{int}A$ , which implies  $A$  is open.  $\square$

**Definition 2.13** ( $F_\sigma$  sets,  $G_\delta$  sets). A countable union of closed sets is called an  $F_\sigma$  set. The complement of an  $F_\sigma$  set, i.e. a countable intersection of open sets, is called a  $G_\delta$  set.

Clearly every closed set is an  $F_\sigma$  set and similarly every open set is a  $G_\delta$  set. Since the countable union of a countable union is again a countable union, a countable union of  $F_\sigma$  sets is an  $F_\sigma$  set. Also, it is quite easy to verify, that a finite intersection of  $F_\sigma$  sets is also an  $F_\sigma$  set.

**Definition 2.14** (Separation axioms). Let  $j \in \{0, 1, 2, 3, 4\}$ . We say that a space  $X$  is a  $T_j$  space, or more succinctly, that  $X$  is  $T_j$ , if it satisfies the condition  $(T_j)$  below.

- $(T_0)$  If  $a, b \in X$  are distinct points, then at least one of the points  $a, b$  has a neighbourhood not containing the other point.

- $(T_1)$  If  $a, b \in X$  are distinct points, then both of the points  $a, b$  have a neighbourhood not containing the other point.
- $(T_2)$  If  $a, b \in X$  are distinct points, the points  $a$  and  $b$  have disjoint neighbourhoods.
- $(T_3)$  If  $a \in X$ , and  $B$  is a closed subset of  $X$  not containing the point  $a$ , then the point  $a$  and the set  $B$  have disjoint neighbourhoods.
- $(T_4)$  If the closed sets  $A, B \subset X$  are disjoint, the sets  $A$  and  $B$  have disjoint neighbourhoods.

Spaces satisfying the condition  $T_2$  are called *Hausdorff* spaces. We call a topological space  $X$  *regular*, if it satisfies the separation axioms  $T_1$  and  $T_3$ . A space is *normal*, if it satisfies axioms  $T_1$  and  $T_4$ . A *completely normal* space is a space whose every subspace is normal. Since the property  $T_1$  is clearly hereditary, in order to prove that a normal space is completely normal, it suffices to show that every subspace of that space is  $T_4$ .

**Proposition 2.15.** *A space  $(X, \mathcal{T})$  is  $T_3$ , if and only if for every  $x \in X$  and for every neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\overline{V} \subset U$ .*

*Proof.* Suppose  $X$  is a  $T_3$ -space. Let  $x \in X$  and  $U \subset X$  be a neighbourhood of  $x$ . Now the set  $\mathbb{C}U$  is closed, and does not contain the point  $x$ . Since  $X$  is  $T_3$ ,  $x$  and  $\mathbb{C}U$  have disjoint neighbourhoods, that we shall call  $V$  and  $W$ , respectively. Now the set  $\mathbb{C}W$  is closed in  $X$  and contains  $V$ , whence it follows that  $\overline{V} \subset \mathbb{C}W \subset \mathbb{C}\mathbb{C}U = U$ .

Suppose the condition of the proposition holds. Let  $x \in X$  and  $B \subset X$  be a closed set not containing the point  $x$ . Now  $\mathbb{C}B$  is a neighbourhood of  $x$ , and thus there exists some neighbourhood  $V$  of  $x$  such that  $\overline{V} \subset \mathbb{C}B$ . Now we have found disjoint neighbourhoods for the point  $x$  and closed set  $B$ , namely  $V$  and  $\mathbb{C}\overline{V}$ , respectively.  $\square$

**Proposition 2.16.** *A space  $(X, \mathcal{T})$  is  $T_4$ , if and only if for every closed subset  $A \subset X$  and for every neighbourhood  $U$  of  $A$ , there exists a neighbourhood  $V$  of  $A$  such that  $\overline{V} \subset U$ .*

*Proof.* The argument is essentially the same as in the proof of Proposition 2.15. We just replace the point  $x$  with the set  $A$ .  $\square$

**Proposition 2.17.** *A space  $(X, \mathcal{T})$  is completely normal, if and only if there exist disjoint neighbourhoods for any two subsets  $A, B \subset X$ , for which it holds that  $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ .*

*Proof.* Suppose  $(X, \mathcal{T})$  is completely normal and that  $A, B \subset X$  are subsets of  $X$  such that  $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ . Let  $Y = \mathcal{C}(\text{cl}_X A \cap \text{cl}_X B)$ . Now  $\text{cl}_Y A$  and  $\text{cl}_Y B$  are disjoint closed subsets of the space  $Y$ . Since  $Y$  is normal and thus  $T_4$ , there exist disjoint neighbourhoods  $U_A, U_B \subset Y$  for the sets  $\text{cl}_Y A$  and  $\text{cl}_Y B$ , respectively. Since  $Y$  is open in  $X$ , so are the sets  $U_A$  and  $U_B$ . Since

$$Y = \mathcal{C}(\text{cl}_X A \cap \text{cl}_X B) = \mathcal{C}(\text{cl}_X A) \cup \mathcal{C}(\text{cl}_X B),$$

both  $A \subset \mathcal{C}(\text{cl}_X B)$  and  $B \subset \mathcal{C}(\text{cl}_X A)$  are subsets of  $Y$ . Hence we have found two disjoint neighbourhoods for  $A$  and  $B$  in  $X$ , namely  $U_A$  and  $U_B$ , respectively.

Suppose the condition of the proposition holds, and that  $Y$  is a subspace of the space  $(X, \mathcal{T})$ . Let  $A, B \subset Y$  be disjoint and closed in  $Y$ . Since the topology on  $Y$  is the subspace topology, we have  $A = Y \cap \text{cl}_X A$  and  $B = Y \cap \text{cl}_X B$ . Thus

$$\emptyset = A \cap B = (Y \cap \text{cl}_X A) \cap B = (\text{cl}_X A) \cap B,$$

and similarly

$$\emptyset = A \cap B = A \cap (Y \cap \text{cl}_X B) = A \cap (\text{cl}_X B).$$

By assumption, there exist disjoint open subsets  $U_A$  and  $U_B$  of  $X$  such that  $A \subset U_A$  and  $B \subset U_B$ . Taking  $U_A \cap Y$  and  $U_B \cap Y$ , we have disjoint neighbourhoods in the subspace topology on  $Y$  for  $A$  and  $B$ , respectively. This shows that  $Y$  is  $T_4$ , completing the proof.  $\square$

**Proposition 2.18** (Urysohn's lemma). *For every pair  $A, B$  of disjoint closed subsets of a normal space  $X$ , there exists a continuous function  $f: X \rightarrow [0, 1]$  such that  $f(x) = 0$  for every  $x \in A$  and  $f(x) = 1$  for every  $x \in B$ .*

*Proof.* For every rational number  $q$  on the interval  $[0, 1]$  we shall define an open set  $V_q$  with the conditions

$$\overline{V}_q \subset V_{q'} \quad \text{whenever} \quad q < q', \quad (1)$$

and

$$A \subset V_0, \quad B \subset X \setminus V_1. \quad (2)$$

We shall define the sets  $V_q$  inductively. Let us arrange all rational numbers on the interval  $]0, 1[$  into an infinite sequence  $q_3, q_4, \dots$  and let  $q_1 = 0$  and  $q_2 = 1$ . Put  $V_0 = U$  and  $V_1 = X \setminus B$ , where  $U$  is an open set satisfying  $A \subset U \subset \overline{U} \subset X \setminus B$ . The existence of  $U$  is guaranteed by Proposition 2.16. Thus  $\overline{V}_0 \subset V_1$ . Condition (2) as well as the condition

$$\overline{V}_{q_i} \subset V_{q_j}, \quad \text{whenever} \quad q_i < q_j \quad \text{and} \quad i, j \leq k \quad (3_k)$$

are thus satisfied for  $k = 2$ .

Suppose the sets  $V_{q_i}$  satisfying  $(3_n)$  are defined for  $i \leq n$ , where  $n \geq 2$ . Let us denote by  $q_l$  and  $q_r$ , respectively, those of the numbers  $q_1, q_2, \dots, q_n$  that are closest to the number  $q_{n+1}$  from the left and from the right. Since  $q_l < q_r$  it follows from  $(3_n)$  that  $\overline{V_{q_l}} \subset V_{q_r}$ . Let  $U'$  be an open set such that  $\overline{V_{q_l}} \subset U' \subset \overline{U'} \subset V_{q_r}$ . Again the existence of such a set  $U'$  follows from Proposition 2.16. Taking  $V_{q_{n+1}} = U'$ , we obtain sets  $V_{q_1}, V_{q_2}, \dots, V_{q_{n+1}}$  which satisfy  $(3_{n+1})$ . The sequence  $V_{q_1}, V_{q_2}, \dots$  obtained this way satisfies conditions (1) and (2).

Consider the function  $f: X \rightarrow [0, 1]$  defined by the formula

$$f(x) = \begin{cases} \inf\{q: x \in V_q\}, & \text{for } x \in V_1 \\ 1, & \text{for } x \in X \setminus V_1 \end{cases}$$

Since (2) implies  $fA \subset \{0\}$  and  $fB \subset \{1\}$ , we only need to show that  $f$  is continuous. To show this, it suffices to show that inverse images of intervals of the form  $[0, a[$  and  $]b, 1]$ , where  $a \leq 1$  and  $b \geq 0$ , are open. The inequality  $f(x) < a$  holds, if and only if there exists a  $q < a$  such that  $x \in V_q$ . Hence the set

$$f^{-1}[0, a[ = \bigcup \{V_q: q < a\}$$

is open as a union of open sets. The inequality  $f(x) > b$  holds, if and only if there exists a  $q' > b$  such that  $x \notin V_{q'}$ . Hence by (1) this means that there exists a  $q > b$  such that  $x \notin \overline{V_q}$ . Thus the preimage

$$f^{-1}]b, 1] = \bigcup \{X \setminus \overline{V_q}: q > b\} = X \setminus \bigcap \{\overline{V_q}: q > b\}$$

is open as a complement of a closed set.  $\square$

Since the unit interval  $[0, 1]$  is homeomorphic to any interval of the form  $[a, b]$ , where  $a < b$  we can substitute the unit interval with any other closed interval in Urysohn's lemma.

**Definition 2.19** (Countability axioms). A space  $X$  is said to have a countable basis at the point  $x$ , if there exists a countable collection  $\mathcal{B}_x$  of neighbourhoods of  $x$  such that each neighbourhood of  $x$  contains some element of  $\mathcal{B}_x$ . A space that has a countable basis at each of its points is said to be *first-countable*. If a space  $X$  has a countable basis for its topology, then  $X$  is said to be *second-countable*.

**Definition 2.20** (Compact space). A space  $X$  is *compact*, if every open covering of  $X$  contains a finite subcovering.

We recall that the set  $A$  is countable, if there exists a surjection  $f: \mathbb{N} \rightarrow A$ .

**Definition 2.21** (Lindelöf space). A space  $X$  is *Lindelöf*, if every open covering of  $X$  contains a countable subcovering.

**Proposition 2.22.** *Every second-countable space is Lindelöf.*

*Proof.* Let  $(X, \mathcal{T})$  be a topological space and let  $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$  be a basis for the topology  $\mathcal{T}$ . Suppose  $\mathcal{D}$  is an open covering of  $X$ . For every  $B_n \in \mathcal{B}$  that is a subset of some element of  $\mathcal{D}$ , we shall choose some  $U_n$  such that  $B_n \subset U_n \in \mathcal{D}$ . If no such member of  $\mathcal{D}$  exists for some basis set  $B_n$ , the set  $U_n$  is not defined. Let us denote the collection of the sets  $U_n$  as  $\mathcal{A}$ . The collection  $\mathcal{A}$  is a countable subcollection of  $\mathcal{D}$ .

Suppose  $x \in X$ . Because  $\mathcal{D}$  is a covering of  $X$ , there exists some  $U \in \mathcal{D}$  such that  $x \in U$ . Because  $\mathcal{B}$  is a basis for  $\mathcal{T}$ , Proposition 2.7 implies the existence of some  $B_n \in \mathcal{B}$  such that  $x \in B_n \subset U$ . Thus there exists a set  $U_n$  for which it holds that  $x \in B_n \subset U_n \in \mathcal{A}$ . Thus  $\mathcal{A}$  is a covering of  $X$ .  $\square$

**Definition 2.23** (Separation, Connected space). Let  $(X, \mathcal{T})$  be a topological space. A *separation* of  $X$  is a pair  $U, V$  of non-empty subsets  $X$  such that  $\overline{U} \cap V = U \cap \overline{V} = \emptyset$ , and  $X = U \cup V$ . The space  $X$  is said to be *connected*, if no separation of  $X$  exists.

The definition of connectedness can also be formulated in the following way: the space  $X$  is connected, if and only if there exist no other subsets of  $X$ , which are both open and closed in  $X$ , except the empty set and the whole space  $X$  itself.

**Definition 2.24** (Separator between subsets). If  $A_1$  and  $A_2$  are disjoint subsets of the space  $(X, \mathcal{T})$ , we say that a subset  $B \subset X$  is a *separator between*  $A_1$  and  $A_2$ , if  $X \setminus B$  can be split into two disjoint sets  $A'_1$  and  $A'_2$ , open in  $X \setminus B$ , and containing  $A_1$  and  $A_2$ , respectively. In other words, it holds that

$$\begin{aligned} X \setminus B &= A'_1 \cup A'_2, \\ A_1 &\subset A'_1, \quad A_2 \subset A'_2, \\ A'_1 \cap A'_2 &= \emptyset, \end{aligned}$$

with  $A'_1$  and  $A'_2$  both open in  $X \setminus B$ , or equivalently both closed in  $X \setminus B$ .

**Proposition 2.25.** *The empty set is a separator between the subsets  $A_1$  and  $A_2$  of a topological space  $(X, \mathcal{T})$ , if and only if there exists a set  $A'_1$  such that*

$$\begin{aligned} A_1 &\subset A'_1, \\ A'_1 \cap A_2 &= \emptyset, \end{aligned}$$

*and  $A'_1$  is both open and closed, or equivalently, has an empty boundary.*



*Proof.* Take  $A'_2 = X \setminus A'_1$ . □

**Definition 2.26** (Dense set). A subset  $D$  of a topological space  $(X, \mathcal{T})$  is called *dense* in  $X$ , if for every  $\emptyset \neq U \in \mathcal{T}$  it holds that the intersection  $D \cap U$  is not empty.

For example the set of rational numbers  $\mathbb{Q}$  is dense in the real numbers  $\mathbb{R}$ , when  $\mathbb{R}$  is endowed with the usual topology. As the set of rational numbers is countable, we get that  $\mathbb{R}$  is in fact a separable space, which leads us to our next definition.

**Definition 2.27** (Separable space). A space  $X$  is called *separable*, if it contains a countable dense set.

## Metric spaces

**Definition 2.28** (Metric). Let  $X$  be some set. A function  $d: X \times X \rightarrow \mathbb{R}_{\geq 0}$  is a *metric* in  $X$ , if the following conditions hold for all  $x, y, z \in X$ :

- (M1)  $d(x, z) \leq d(x, y) + d(y, z)$ ,
- (M2)  $d(x, y) = d(y, x)$ ,
- (M3)  $d(x, y) = 0$ , if and only if  $x = y$ .

A metric space is an ordered pair  $(X, d)$ , where  $X$  is some set and  $d$  a metric in  $X$ . The following definition gives us a tool to define a topology on a metric space.

**Definition 2.29** (Ball). Suppose  $a$  is an element of a metric space  $(X, d)$ , and  $r > 0$ . We define

$$\begin{aligned} B(a, r) &:= \{x \in X : d(x, a) < r\} \\ \overline{B}(a, r) &:= \{x \in X : d(x, a) \leq r\} \\ S(a, r) &:= \{x \in X : d(x, a) = r\}. \end{aligned}$$

We call the set  $B(a, r)$  an *open ball* with centre  $a$  and radius  $r$ . The set  $\overline{B}(a, r)$  is called a *closed ball*, and  $S(a, r)$  is called a *sphere*. In the Euclidean space  $\mathbb{R}^n$  we usually write  $B^n(a, r)$  for open balls, and  $\overline{B}^n(a, r)$  for closed balls. For the sphere we write  $S^{n-1}(a, r)$ . Why the dimension of the sphere is decremented by one will be explained later.

**Definition 2.30** (Metric topology). Let  $(X, d)$  be a metric space. The collection of balls  $\{B(x, r) : x \in X, r > 0\}$  is a basis for a topology on  $X$ , called the *metric topology* induced by  $d$ .

In a metric space  $(X, d)$  a set  $U \subset X$  is open, if for every  $x \in U$  we can find an open ball  $B(x, r)$ , which is included in  $U$ .

**Definition 2.31** (Diameter). The *diameter* of a subset  $A$  of a metric space  $(X, d)$  is defined to be

$$d(A) := \sup\{d(x, y) : x, y \in A\}.$$

The *distance* of a point  $x \in X$  from the set  $A$  is

$$d(x, A) := \inf\{d(x, y) : y \in A\}.$$

**Proposition 2.32.** *If  $(X, d)$  is a separable metric space, it is second-countable.*

*Proof.* Let  $(X, d)$  be a metric space and  $A = \{a_j\}_{j \in \mathbb{N}}$  be a countable dense set in  $X$ . We claim that the collection

$$\mathcal{B} := \{B(a, r) : a \in A, r \in \mathbb{Q}_{>0}\}$$

is a countable basis for the topology on  $X$ .

Let  $U$  be a non-empty open set in  $X$ . Now for every  $x \in U$ , there exists some  $\varepsilon > 0$  such that the ball  $B(x, \varepsilon)$  is included in  $U$ . Since  $A$  is dense in  $X$ , there exists some  $a_j \in A$  such that  $d(x, a_j) < \varepsilon/2$ . Hence,  $a_j \in B(x, \varepsilon) \subset U$ . It holds that  $B(a_j, \varepsilon/2) \subset B(x, \varepsilon)$ . For let  $y \in B(a_j, \varepsilon/2)$ . Now from the triangle inequality we get that

$$d(y, x) \leq d(y, a_j) + d(a_j, x) < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , for every  $\varepsilon > 0$  we can find a positive rational number  $r$  such that  $r < \varepsilon$ . Thus for every  $x \in U$ , we can find an open ball with centre at some  $a_j \in A$  and a rational radius  $r$  such that

$$x \in B(a_j, r) \subset B(x, \varepsilon) \subset U.$$

Hence we have the inclusion

$$U \subset \bigcup_{a \in U \cap A} B(a, r_a),$$

where  $r_a$  is a rational radius depending on the point  $a$ , such that the ball  $B(a, r_a)$  is included in  $U$ . The condition that the balls  $B(a, r_a)$ , where  $a \in A$  and  $r \in \mathbb{Q}_{>0}$ , are included in  $U$  gives us the other inclusion

$$\bigcup_{a \in U \cap A} B(a, r_a) \subset U.$$

We have shown that every non-empty open subset of  $X$  can be represented as a union of sets in the collection  $\mathcal{B}$ . This completes the proof, since  $\mathcal{B}$  is countable.  $\square$

**Proposition 2.33** (The Lebesgue Covering Theorem). *For every open covering  $\mathcal{D}$  of a compact metric space  $(X, d)$ , there exists an  $\varepsilon > 0$  such that the covering  $\{B(x, \varepsilon)\}_{x \in X}$  is a refinement of  $\mathcal{D}$ . In other words, every set in  $\{B(x, \varepsilon)\}_{x \in X}$  is contained in some set in  $\mathcal{D}$ .*

*Proof.* For every  $x \in X$ , we shall choose some  $\varepsilon_x > 0$  such that the ball  $B(x, 2\varepsilon_x)$  is contained in a member of  $\mathcal{D}$ . Since  $X$  is compact, the open covering  $\{B(x, \varepsilon)\}_{x \in X}$  has a finite subcovering, i.e. there exists a finite set  $\{x_1, x_2, \dots, x_k\} \subset X$  such that

$$X = \bigcup_{i=1}^k B(x_i, \varepsilon_{x_i}).$$

The number  $\varepsilon := \min\{\varepsilon_{x_1}, \varepsilon_{x_2}, \dots, \varepsilon_{x_k}\}$  has the required property.  $\square$

**Definition 2.34** (Hilbert cube). The space  $I_\omega := \prod_{i \in \mathbb{N}} [-1/i, 1/i]$  is called the *Hilbert cube*. Since  $[-1/i, 1/i]$  is compact for all  $i \in \mathbb{N}$ , the Hilbert cube is compact by Väisälä [7, Lause 18.1, p. 136]. It is often convenient to think of the Hilbert cube as a metric space. For this purpose, the Hilbert cube is considered to be a subspace of the separable metric space  $l^{2*}$ . The  $l^2$ -norm induces the product topology on the Hilbert cube.

**Definition 2.35** (Metrizable space). A space  $(X, \mathcal{T})$  is called *metrizable*, if there exists a metric  $d$ , such that the topology  $\mathcal{T}_d$  induced by this metric is equal to  $\mathcal{T}$ .

We recall, that every metric space is Hausdorff, regular and normal, see for example Väisälä [7, Havaintoja, p. 87]. In fact, every metric space is completely normal, since every subspace of a metric space is itself a metric space.

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\*Here we mean the space consisting of square-summable sequences of real numbers. The  $l^2$ -norm is naturally defined as  $\|x\|_2 := (\sum_{i \in \mathbb{N}} |x_i|^2)^{1/2}$ .



## Small Inductive Dimension

We shall begin this chapter by examining 0-dimensional spaces and their properties.

### Dimension 0

**Definition 3.1** (Dimension 0). A non-empty regular space  $X$  has dimension 0, if for every point  $x \in X$  and for every neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $V \subset U$  and  $\partial V = \emptyset$ .

**Proposition 3.2.** *A 0-dimensional space can also be defined as a non-empty space, in which we can define a basis consisting of sets that are both open and closed.*

*Proof.* Let  $X$  be some non-empty regular space, which has a basis  $\mathcal{B}$  consisting of open and closed sets. Let  $x \in X$  and  $U$  be a neighbourhood of  $x$ . Now Proposition 2.7 guarantees, that we can find a basis element  $B \in \mathcal{B}$  such that  $x \in B \subset U$ . This basis set  $B$  is a neighbourhood of  $x$ , which is both open and closed, and by Proposition 2.12 has an empty boundary.

Suppose then, that  $X$  is a 0-dimensional space. Now for every point  $x \in X$  and for every neighbourhood  $V$  of  $x$ , we can find a new neighbourhood  $U \subset V$  of  $x$ , whose boundary is empty, or equivalently is both open and closed. Since we can find arbitrarily small open and closed neighbourhoods for each point in  $X$ , Proposition 2.7 implies the collection of open and closed sets in  $X$  form a basis for the topology on  $X$ .  $\square$

**Example 3.3.** Every non-empty countable metric space  $(X, d)$  is 0-dimensional. For suppose that  $x \in X$  and  $U$  is some neighbourhood of  $x$ . Let  $r > 0$  be such, that the ball  $B(x, r)$  is contained in  $U$ . Let  $x_i$ , where  $i \in \mathbb{N}$

be an enumeration of  $X$ . Let  $r'$  be a positive real number less than  $r$  such that  $r' \neq d(x_i, x)$  for all  $i \in \mathbb{N}$ . Now the ball  $B(x, r')$  is contained in  $U$ , and has an empty boundary. Hence  $X$  is 0-dimensional.

**Example 3.4.** Any non-empty subspace of the real line  $\mathbb{R}$  containing no interval is 0-dimensional. For suppose that  $P$  is such a set. The complement  $\mathbb{R} \setminus P$  is dense in  $\mathbb{R}$ , for otherwise there would be some non-empty open set  $U$  which does not intersect  $\mathbb{R} \setminus P$ . Hence we would have  $U \subset P$ . Since  $U$  is open in  $\mathbb{R}$ , it contains an interval, a contradiction. Thus sets of the form

$$]a, b[ \cap P, \quad (*)$$

where  $a, b \in \mathbb{R} \setminus P$  and  $a < b$  form a basis for the subspace topology on  $P$ . Since the complement in the subspace  $P$  of a set of the form  $(*)$  is

$$P \cap (]-\infty, a[ \cup ]b, \infty]) = P \cap (]-\infty, a[ \cup ]b, \infty[),$$

which is open in  $P$ , the topology on  $P$  has a basis consisting of open and closed sets. Hence by Proposition 3.2  $P$  is 0-dimensional.

**Example 3.5.** The subspace  $\mathcal{J}_\omega$  of the Hilbert cube  $I_\omega$  consisting of points, all of whose coordinates are irrational is 0-dimensional. Suppose  $a \in \mathcal{J}_\omega$ , and let  $U$  be a neighbourhood of  $a$  in  $I_\omega$ . Since the Hilbert cube has the product topology, there exists a number  $N \in \mathbb{N}$  such that  $\text{pr}_i U = [-1/i, 1/i]$  for  $i > N$ . Thus we can find a new neighbourhood for  $a$  contained in  $U$ , which consists of the points  $x = (x_1, x_2, \dots)$  in  $I_\omega$  whose first  $N$  coordinates are restricted by  $p_i < x_i < q_i$ , where  $p_i < a_i < q_i$ , and  $p_i, q_i$  are sufficiently close to  $a_i$ , and the rest of the coordinates of  $x$  are restricted only by  $|x_i| \leq 1/i$ .

By taking  $p_i$  and  $q_i$  rational, we get a neighbourhood  $V$  of  $a$ , each of whose boundary point in  $I_\omega$  has at least one rational coordinate. Thus  $V$  has an empty boundary in  $\mathcal{J}_\omega$ , which proves that  $\mathcal{J}_\omega$  is 0-dimensional.

**Proposition 3.6.** *A non-empty subspace of a 0-dimensional space is 0-dimensional.*

*Proof.* Suppose  $X$  is a 0-dimensional space and let  $X'$  be a non-empty subspace of  $X$ . Let  $x \in X$  and let  $U'$  be a neighbourhood of the point  $x$  in the subspace  $X'$ . Since  $X'$  has the subspace topology, there exists some  $U$  open in  $X$  such that  $U' = U \cap X'$ .

Since  $X$  is 0-dimensional, there exists a neighbourhood  $V$  of  $x$ , both open and closed in  $X$  such that  $V \subset U$ . Let  $V' = V \cap X'$ . Now  $V'$  is both open and closed in  $X'$ , and  $x \in V' \subset U'$ . Hence  $X'$  is 0-dimensional.  $\square$

We shall prove that Definition 3.1 is equivalent to

**Proposition 3.7.** *A non-empty regular space  $X$  has dimension 0, if and only if the empty set is a separator between every singleton set  $\{x\} \subset X$  and every closed set  $C$  not containing the point  $x$ .*

*Proof.* Suppose  $X$  is 0-dimensional. Suppose  $x \in X$  and let  $C \subset X$  be a non-empty closed subset of  $X$  not containing the point  $x$ . Since  $X \setminus C$  is a neighbourhood of  $x$ , there exists a neighbourhood  $V \subset X \setminus C$  of  $x$ , which is both open and closed. From  $V \cap C = \emptyset$  and Proposition 2.25, it follows that the empty set is a separator between  $\{x\}$  and  $C$ .

Suppose then, that the condition of the proposition holds. Suppose  $x \in X$  and that  $V$  is a neighbourhood of  $x$ . Now the set  $\mathbb{C}V$  is a closed subset of  $X$  that does not contain the point  $x$ . Since the empty set is a separator between  $\{x\}$  and  $\mathbb{C}V$ , Proposition 2.25 implies that there exists an open and closed set  $U$  such that  $x \in U$ , and  $U \cap \mathbb{C}V = \emptyset$ . Hence we have found a neighbourhood  $U \subset V$  of  $x$ , whose boundary is empty. This proves the claim.  $\square$

From Proposition 3.7 it follows that a metric space is 0-dimensional, if the empty set is a separator between any two disjoint closed subsets, since in metric spaces singleton sets are closed. The converse also holds:

**Proposition 3.8.** *If the separable metric space  $X$  is 0-dimensional, the empty set is a separator between any two disjoint closed subsets of  $X$ .*

*Proof.* Suppose  $(X, \mathcal{T})$  is 0-dimensional. Proposition 3.7 implies that the empty set is a separator between any singleton set  $\{x\} \subset X$  and any closed set not containing  $x$ . Let  $C$  and  $K$  be disjoint closed subsets of  $X$ . We shall demonstrate that the empty set is a separator between these two sets in  $X$ .

For every point  $x \in X$ , at most one of the conditions  $x \in C$  and  $x \in K$  holds. Since  $X$  is  $T_3$ , there exist open and closed neighbourhoods  $U_x$  for each point  $x$  such that again at most one of the following hold:  $U_x \cap C \neq \emptyset$ , or  $U_x \cap K \neq \emptyset$ . By Proposition 2.32  $X$  is second-countable and by Proposition 2.22  $X$  is Lindelöf. Hence there exists a countable collection  $\{U_{x_n} : n \in \mathbb{N}\}$  of these neighbourhoods  $U_x$  that forms an open covering of  $X$ . We shall define a new sequence of sets as follows:

$$V_1 := U_{x_1},$$

$$V_i := U_{x_i} \setminus \left( \bigcup_{k=1}^{i-1} U_{x_k} \right), \quad i = 2, 3, 4, \dots$$

Now we have

$$X = \bigcup_{i=1}^{\infty} V_i,$$

and

$$V_i \cap V_j = \emptyset, \quad \text{if } i \neq j.$$

Since  $\bigcup_{k=1}^{i-1} U_{x_k}$  is closed for every  $i \in \mathbb{N}_{>1}$  as the finite union of closed sets, its complement is open, and thus  $V_i$  is open as the intersection of two open sets. We also have either  $V_i \cap C = \emptyset$  or  $V_i \cap K = \emptyset$ , (or both).

Let  $C'$  be the union of all  $V_i$  for which  $V_i \cap K = \emptyset$ , and  $K'$  the union of the remaining  $V_i$ . Now

$$\begin{aligned} X &= C' \cup K', \\ C' \cap K' &= \emptyset, \end{aligned}$$

both  $C'$  and  $K'$  are open as unions of open sets, and

$$C' \cap K = K' \cap C = \emptyset.$$

It follows that  $C \subset C'$  and  $K \subset K'$ . We have shown that the empty set is a separator between  $C$  and  $K$ .  $\square$

**Proposition 3.9.** *If  $C_1$  and  $C_2$  are disjoint closed subsets of a separable metric space  $X$ , and  $A$  is a 0-dimensional subspace of  $X$ , there exists a closed separator  $B$  between the sets  $C_1$  and  $C_2$  such that  $A \cap B = \emptyset$ .*

*Proof.* Let the sets  $C_1$ ,  $C_2$  and  $A$  be as described in the proposition. Since  $X$  is  $T_4$ , Proposition 2.16 implies there exist open sets  $U_1$  and  $U_2$  such that

$$C_1 \subset U_1, \quad C_2 \subset U_2, \tag{1}$$

and  $\overline{U_1} \cap \overline{U_2} = \emptyset$ .

The disjoint sets  $\overline{U_1} \cap A$  and  $\overline{U_2} \cap A$  are closed in  $A$ , and by Proposition 3.8 the empty set is a separator between these sets in  $A$ , since  $A$  is 0-dimensional. Thus there exist disjoint sets  $C'_1$  and  $C'_2$ , both open and closed in  $A$ , satisfying

$$A = C'_1 \cup C'_2$$

and

$$\overline{U_1} \cap A \subset C'_1, \quad \overline{U_2} \cap A \subset C'_2.$$

Thus

$$C'_1 \cap \overline{U_2} = C'_2 \cap \overline{U_1} = \emptyset, \tag{2}$$

and

$$C_1 \cap \overline{C'_2} = \overline{C'_1} \cap C'_2 = \emptyset. \tag{3}$$

From (1) and (2) we get

$$C'_1 \cap \overline{C_2} = C'_2 \cap \overline{C_1} = \emptyset. \tag{4}$$



Furthermore, since  $U_1$  and  $U_2$  are open sets, (2) implies

$$\overline{C'_1} \cap U_2 = \overline{C'_2} \cap U_1 = \emptyset,$$

and hence by (1)

$$\overline{C'_1} \cap C_2 = \overline{C'_2} \cap C_1 = \emptyset. \quad (5)$$

From (3), (4), and (5) together with  $\overline{C_1} \cap \overline{C_2} = C_1 \cap C_2 = \emptyset$  it follows that  $\overline{C_1 \cup C'_1} \cap (C_2 \cup C'_2) = (C_1 \cup C'_1) \cap \overline{C_2 \cup C'_2} = \emptyset$ . Since  $X$  is completely normal, Proposition 2.17 implies there exists an open set  $V$  such that

$$C_1 \cup C'_1 \subset V,$$

and

$$\overline{V} \cap (C_2 \cup C'_2) = \emptyset.$$

Now we have

$$\begin{aligned} X \setminus \partial V &= V \cup (\mathbb{C}V \setminus \partial V), \\ C_1 &\subset V, \quad C_2 \subset (\mathbb{C}V \setminus \partial V), \end{aligned}$$

the sets  $V$  and  $\mathbb{C}V \setminus \partial V$  are open and closed in  $X \setminus \partial V$ , and the boundary  $\partial V$  is disjoint from  $C'_1 \cup C'_2 = A$ . We have shown that we can choose  $B = \partial V$ , which completes the proof.  $\square$

**Theorem 3.10** (The Union Theorem for Dimension 0). *If the separable metric space  $X$  is the countable union of closed 0-dimensional subsets of  $X$ , then the space  $X$  is itself 0-dimensional.*

*Proof.* Suppose

$$X = \bigcup_{i \in \mathbb{N}} C_i,$$

where each  $C_i$  is closed and 0-dimensional. Let  $K$  and  $L$  be disjoint closed subsets of  $X$ . We will show that the empty set is a separator between these sets. Now  $K \cap C_1$  and  $L \cap C_1$  are disjoint closed subsets of the 0-dimensional space  $C_1$ . Thus by Proposition 3.8 there exist subsets  $A_1$  and  $B_1$  of  $C_1$ , which are closed in  $C_1$ , and thus also closed in  $X$ , such that

$$K \cap C_1 \subset A_1, \quad L \cap C_1 \subset B_1,$$

$$A_1 \cup B_1 = C_1, \quad \text{and} \quad A_1 \cap B_1 = \emptyset.$$

The sets  $K \cup A_1$  and  $L \cup B_1$  are closed and disjoint in  $X$ . Since  $X$  is normal, there exist open sets  $G_1$  and  $H_1$  such that

$$K \cup A_1 \subset G_1, \quad L \cup B_1 \subset H_1,$$

and

$$\overline{G}_1 \cap \overline{H}_1 = \emptyset.$$

Hence it holds that

$$C_1 \subset G_1 \cup H_1,$$

and

$$K \subset G_1, \quad L \subset H_1.$$

We shall repeat this same process, replacing  $C_1$  by  $C_2$ , and replacing  $K$  and  $L$  by  $\overline{G}_1$  and  $\overline{H}_1$ , respectively. We get open sets  $G_2$  and  $H_2$  for which

$$C_2 \subset G_2 \cup H_2,$$

$$\overline{G}_1 \subset G_2, \quad \overline{H}_1 \subset H_2,$$

and

$$\overline{G}_2 \cap \overline{H}_2 = \emptyset.$$

We shall continue inductively, constructing sequences  $\{G_i\}_{i \in \mathbb{N}}$  and  $\{H_i\}_{i \in \mathbb{N}}$  of sets open in  $X$  for which it holds that

$$C_i \subset G_i \cup H_i,$$

$$\overline{G}_{i-1} \subset G_i, \quad \overline{H}_{i-1} \subset H_i,$$

and

$$\overline{G}_i \cap \overline{H}_i = \emptyset.$$

Put

$$G := \bigcup_{i \in \mathbb{N}} G_i \quad \text{and} \quad H := \bigcup_{i \in \mathbb{N}} H_i.$$

Now  $G$  and  $H$  are disjoint open sets such that

$$X = \bigcup_{i \in \mathbb{N}} C_i \subset G \cup H$$

and

$$K \subset G, \quad L \subset H.$$

This shows that the empty set is a separator between  $K$  and  $L$ , which completes the proof.  $\square$

**Corollary 3.11.** *If a separable metric space  $X$  can be represented as a countable union of 0-dimensional  $F_\sigma$  sets, then  $X$  is itself 0-dimensional.*

*Proof.* Suppose  $X = \cup_{i \in \mathbb{N}} C_i$ , where each  $C_i$  is a 0-dimensional  $F_\sigma$  set. Now each of the sets  $C_i$  is a countable union of closed sets, which are at most 0-dimensional by Theorem 3.6. The countable union of a countable union is itself a countable union, whence it follows that  $X$  can be represented as the countable union of 0-dimensional closed subsets. Hence by Theorem 3.10  $X$  is 0-dimensional.  $\square$

**Example 3.12.** Suppose  $m \geq 0$  and denote by  $\mathcal{Q}_\omega^m$  the set of points of the Hilbert cube exactly  $m$  of whose coordinates are rational. The set  $\mathcal{Q}_\omega^m$  is 0-dimensional.

For each selection of  $m$  indexes  $i_1, i_2, \dots, i_m$ , and for each selection of  $m$  rational numbers  $r_1, r_2, \dots, r_m$  we have a subspace of the Hilbert cube defined by the equations

$$x_{i_1} = r_1, \quad x_{i_2} = r_2, \quad \dots, \quad x_{i_m} = r_m.$$

There are countably many of these subspaces. Denote by  $C_i$  the set consisting of the points satisfying the above equations, and all of whose remaining coordinates are irrational. The sets  $C_i$  contain all of their limit points and are thus closed in  $\mathcal{Q}_\omega^m$ . Every  $C_i$  can be embedded into  $\mathcal{J}_\omega$  (Example 3.5). Therefore the sets  $C_i$  are 0-dimensional by Theorem 3.15, which is proved below. Thus  $\mathcal{Q}_\omega^m$  is the countable union of 0-dimensional spaces, and by Theorem 3.10 is itself 0-dimensional.

## Dimension $n$

**Definition 3.13** (Small inductive dimension). Let  $X$  be a regular space. The *small inductive dimension* of  $X$ , denoted by  $\text{ind}(X)$ , is either an integer larger than  $-2$ , or  $\infty$ . The definition of the dimension function  $\text{ind}$  consists of the following conditions:

- (SID1)  $\text{ind}(X) = -1$ , if and only if  $X = \emptyset$ ,
- (SID2)  $\text{ind}(X) \leq n$ , where  $n \in \mathbb{Z}_{\geq 0}$ , if for every point  $x \in X$  and for each neighbourhood  $V \subset X$  of the point  $x$  there exists a neighbourhood  $U \subset V$  of  $x$  such that

$$\text{ind}(\partial U) \leq n - 1,$$

- (SID3)  $\text{ind}(X) = n$  if  $\text{ind}(X) \leq n$  and  $\text{ind}(X) > n - 1$ ,
- (SID4)  $\text{ind}(X) = \infty$  if  $\text{ind}(X) > n$  for all  $n \in \mathbb{Z}_{\geq -1}$

We shall refer to the small inductive dimension simply as dimension throughout this thesis. When we talk about the dimension of some space  $X$ , it is understood that  $X$  is regular.

**Proposition 3.14.** *Equivalent to the condition that  $\text{ind}(X) \leq n$  is the existence of a basis for the topology on  $X$  consisting of sets whose boundaries have dimension  $\leq n - 1$ .*

*Proof.* The argument is essentially the same, as in the proof of Proposition 3.2.  $\square$

**Theorem 3.15.** *Dimension is a topological invariant.*

*Proof.* Throughout the proof suppose that  $X$  and  $Y$  are regular spaces and that  $f: X \approx Y$ .

Suppose that  $X$  is 0-dimensional. Suppose  $y \in Y$  and let  $V$  be a neighbourhood of  $y$ . Now  $f^{-1}V$  is a neighbourhood of the point  $f^{-1}(y)$ . Since  $X$  is 0-dimensional, there exists a neighbourhood  $U \subset f^{-1}V$  of  $f^{-1}(y)$  with an empty boundary. Since homeomorphisms are open and closed maps, the image of  $U$  under  $f$  is open and closed in  $Y$ . We have thus found a neighbourhood  $fU \subset V$  of  $y$  with an empty boundary. Hence  $Y$  is 0-dimensional.

Suppose that dimension is a topological invariant for spaces whose dimension does not exceed some  $n \geq 0$ . Suppose  $\text{ind}(X) = n + 1$ . Now there exists an element  $x \in X$  such that there exists a neighbourhood  $W$  of  $x$  such that all neighbourhoods  $U \subset W$  of  $x$  have boundaries with dimension  $\text{ind}(\partial U) \geq n$ . Thus there exists a point  $y = f(x) \in Y$  with a neighbourhood  $fW$  such that every neighbourhood  $V \subset fW$  of  $y$  has boundaries with dimension  $\text{ind}(\partial V) \geq n$  by the induction hypothesis. Hence  $\text{ind}(Y) \geq n + 1$ .

Suppose that  $y \in Y$  and let  $V'$  be a neighbourhood of  $y$ . Now  $f^{-1}V'$  is a neighbourhood of the point  $f^{-1}(y) \in X$ . Since  $\text{ind}(X) \leq n + 1$  there exists a neighbourhood  $U' \subset f^{-1}V'$  of  $f^{-1}(y)$  whose boundary has dimension  $\text{ind}(\partial U') \leq n$ . The set  $fU' \subset V'$  is a neighbourhood of the point  $y$ . From the induction hypothesis we get that  $\text{ind}(f\partial U') = \text{ind}(\partial fU') \leq n$ . Thus  $\text{ind}(Y) \leq n + 1$ , which proves the claim.  $\square$

**Proposition 3.16.** *For every subspace  $M$  of a regular space  $X$ , we have  $\text{ind}(M) \leq \text{ind}(X)$ .*

*Proof.* If  $\text{ind}(X) = \infty$ , the proposition is evident, so we may suppose  $\text{ind}(X) < \infty$ . The inequality holds clearly, if  $\text{ind}(X) = -1$ . Suppose the theorem holds for all regular spaces, whose dimension does not exceed some  $n \geq -1$ . Suppose  $X$  is a regular space with  $\text{ind}(X) = n + 1$  and that  $M$  is a non-empty subspace of  $X$ . Let  $x \in M$  and  $V_M$  a neighbourhood of  $x$  in  $M$ . Because

the topology on  $M$  is the subspace topology, there exists some  $V_X$  open in  $X$  such that  $V_M = M \cap V_X$ . Since  $\text{ind}(X) \leq n + 1$ , there exists an open set  $U_X \subset X$  such that  $x \in U_X \subset V_X$  and  $\text{ind}(\partial U_X) \leq n$ .

The intersection  $U_M = M \cap U_X$  is open in  $M$  and is included in  $V_M$ . The boundary  $\partial_M U_M$  in the space  $M$  is equal to

$$\partial_M U_M = M \cap \text{cl}_X(M \cap U_X) \cap \text{cl}_X(M \setminus U_X).$$

The boundary  $\partial_M U_M$  is a subspace of the space  $\partial U_X$ . Hence by the induction hypothesis we get that  $\text{ind}(\partial_M U_M) \leq n$ , and thus we have  $\text{ind}(M) \leq n + 1 = \text{ind}(X)$ .  $\square$

**Proposition 3.17.** *Suppose  $\text{ind}(X) = n$ . Then  $X$  contains an  $m$ -dimensional subspace for every  $m \leq n$ .*

*Proof.* It suffices to show that  $X$  contains a subspace with dimension  $n - 1$ . Since  $\text{ind}(X) > n - 1$ , there exists a point  $x \in X$  and a neighbourhood  $V \subset X$  of  $x$  such that for every open set  $U$  satisfying the condition  $x \in U \subset V$ , we have  $\text{ind}(\partial U) \geq n - 1$ . On the other hand, since  $\text{ind}(X) \leq n$ , there exists an open set  $U \subset X$  satisfying the above condition with  $\text{ind}(\partial U) \leq n - 1$ . This proves the claim.  $\square$

**Proposition 3.18.** *A subspace  $X'$  of a separable metric space  $X$  satisfies  $\text{ind}(X') \leq n$ , if and only if every point of the subspace  $X'$  has arbitrarily small neighbourhoods in  $X$ , whose boundaries have intersections with  $X'$  of dimension  $\leq n - 1$ .*

*Proof.* Suppose  $X'$  is a subspace of a separable metric space with dimension  $\text{ind}(X') \leq n$ . Let  $x$  be a point in  $X'$  and  $U$  a neighbourhood of  $x$  in  $X$ . Now

$$U' = U \cap X'$$

is a neighbourhood of  $x$  in  $X'$ . Hence there exists a neighbourhood  $V' \subset X'$  of  $x$  such that

$$x \in V' \subset U',$$

and

$$\text{ind}(\partial_{X'} V') \leq n - 1,$$

where  $\partial_{X'} V'$  denotes the boundary of  $V'$  in the subspace  $X'$ . Now it holds that  $\text{cl}_X V' \cap (X' \setminus \text{cl}_X V') = V' \cap \text{cl}_X (X' \setminus \text{cl}_X V') = \emptyset$ . Since  $X$  is completely normal as a metric space, Proposition 2.17 implies that there exists a set  $W$  open in  $X$  satisfying

$$V' \subset W \quad \text{and} \quad \text{cl}_X W \cap (X' \setminus \text{cl}_X V') = \emptyset.$$

We may assume that  $W \subset U$ , since otherwise we can replace  $W$  by the intersection  $W \cap U$ . The boundary  $\partial_X W = \text{cl}_X W \setminus W$  contains no point of  $V'$  and no point of  $X' \setminus \text{cl}_X V'$ . Hence we have the inclusion  $\partial_X W \cap X' \subset \partial_{X'} V'$  and by Proposition 3.16 we have  $\text{ind}(\partial_X W \cap X') \leq n-1$ , so that the condition of the proposition is satisfied.

Suppose then, that  $X'$  is a subspace of a separable metric space  $X$  for which the condition of the proposition holds. Suppose  $x \in X'$  and let  $U'$  be a neighbourhood of  $x$  in the subspace  $X'$ . Since the topology on  $X'$  is the subspace topology, there exists a neighbourhood  $U$  of  $x$  in  $X$  such that  $U' = U \cap X'$ . Thus there exists a set  $V$ , open in  $X$ , such that

$$x \in V \subset U,$$

and

$$\text{ind}(\partial_X V \cap X') \leq n-1.$$

Let  $V' := V \cap X'$ . Now  $V'$  is open in  $X'$ , and  $x \in V' \subset U'$ . We have the inclusion  $\partial_{X'} V' \subset \partial_X V \cap X'$  from which it follows that  $\text{ind}(\partial_{X'} V') \leq n-1$  by Proposition 3.16, so that  $\text{ind}(X') \leq n$ .  $\square$

**Proposition 3.19.** *For any two subspaces  $A$  and  $B$  of a separable metric space  $X$ , we have*

$$\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B) + 1.$$

*Proof.* The proposition is clear, if

$$\text{ind}(A) = \text{ind}(B) = -1,$$

i.e. the sets  $A$  and  $B$  are empty. Let  $\text{ind}(A) = m$  and  $\text{ind}(B) = n$ , and suppose the proposition holds for the cases

$$\text{ind}(A) \leq m, \quad \text{ind}(B) \leq n-1, \tag{6}$$

and

$$\text{ind}(A) \leq m-1, \quad \text{ind}(B) \leq n. \tag{7}$$

Let  $x \in A \cup B$ . Without loss of generality, we may assume  $x \in A$ . Let  $U$  be a neighbourhood of  $x$  in  $X$ . Now by Proposition 3.18 there exists a set  $V$ , open in  $X$ , such that

$$x \in V \subset U$$

and

$$\text{ind}(\partial_X V \cap A) \leq m-1.$$

As  $\partial_X V \cap B$  is a subset of  $B$ , by Proposition 3.16 we have

$$\text{ind}(\partial_X V \cap B) \leq n.$$

By the induction hypotheses (6) and (7) we get

$$\text{ind}(\partial_X V \cap (A \cup B)) \leq m + n.$$

Hence, from Proposition 3.18 we have

$$\text{ind}(A \cup B) \leq m + n + 1.$$

This completes the proof.  $\square$

**Example 3.20.** In Example 3.4 we showed that a subspace of the real line containing no interval is 0-dimensional. Thus both the rational numbers  $\mathbb{Q}$ , and the set of irrational numbers  $\mathbb{R} \setminus \mathbb{Q}$  are 0-dimensional. We will show later that, in fact,  $\text{ind}(\mathbb{R}) = 1$ . This shows that Proposition 3.19 gives us the most accurate upper bound for the dimension of the union of two subspaces. In case the reader wonders, we have not contradicted Theorem 3.10, since neither  $\mathbb{Q}$ , nor  $\mathbb{R} \setminus \mathbb{Q}$  is a closed subset of  $\mathbb{R}$ .

**Example 3.21.** Suppose  $m \geq 0$ . Denote by  $\mathcal{M}_\omega^m$  the set of points of the Hilbert cube  $I_\omega$  at most  $m$  of whose coordinates are rational. Then  $\text{ind}(\mathcal{M}_\omega^m) \leq m$ . This follows by repeatedly applying Proposition 3.19 since

$$\mathcal{M}_\omega^m = \bigcup_{i=0}^m \mathcal{Q}_\omega^i,$$

where each  $\mathcal{Q}_\omega^i$  is 0-dimensional by Example 3.12.

**Theorem 3.22** (The Union Theorem for Dimension  $n$ ). *Suppose the separable metric space  $X$  is the countable union of  $F_\sigma$  sets of dimension  $\leq n$ . Then  $\text{ind}(X) \leq n$ .*

Suppose the theorem holds for some  $n - 1 \geq -1$ . We shall first prove that this implies the following:

**Corollary 3.23.** *Any separable metric space  $Y$  with dimension  $\text{ind}(Y) \leq n$  is the union of a subspace of dimension  $\leq n - 1$  and a subspace of dimension  $\leq 0$ .*

*Proof.* Let  $Y$  be a separable metric space of dimension  $\leq n$ . Let  $\{B_i : i \in \mathbb{N}\}$  be a basis for the topology on  $Y$ , consisting of sets such that the boundaries have  $\text{ind}(\partial B_i) \leq n - 1$  for all  $i \in \mathbb{N}$ . The existence of such a basis is

guaranteed by Propositions 2.32 and 3.14. Since the boundaries  $\partial B_i$  are closed sets by Proposition 2.11, and are thus  $F_\sigma$  sets, and since Theorem 3.22 was assumed to hold for  $n - 1$ , we get that

$$L := \bigcup_{i \in \mathbb{N}} \partial B_i$$

has dimension  $\leq n - 1$ . Since the boundaries  $\partial B_i$  do not meet  $Y \setminus L$ , the condition of Proposition 3.18 (with  $n = 0$  and  $X' = Y \setminus L$ ) is satisfied. Thus  $\text{ind}(Y \setminus L) \leq 0$ . The claim follows from the equation  $Y = L \cup (Y \setminus L)$ .  $\square$

*Proof of Theorem 3.22.* The case  $n = -1$  is clear. Suppose the theorem holds for some  $n - 1 \geq -1$ . Suppose that

$$X = \bigcup_{i \in \mathbb{N}} C_i,$$

$$\text{ind}(C_i) \leq n,$$

and  $C_i$  is a  $F_\sigma$  set for all  $i \in \mathbb{N}$ . We want to show that  $\text{ind}(X) \leq n$ . Let

$$K_1 := C_1,$$

and let

$$K_i := C_i \setminus \bigcup_{j=1}^{i-1} C_j = C_i \cap \left( X \setminus \bigcup_{j=1}^{i-1} C_j \right), \quad i = 2, 3, 4, \dots$$

Now we have

$$X = \bigcup_{i \in \mathbb{N}} K_i, \tag{8}$$

and

$$K_i \cap K_j = \emptyset, \quad \text{if } i \neq j.$$

Since

$$X \setminus \bigcup_{j=1}^{i-1} C_j = \bigcup_{j=i}^{\infty} C_j$$

is an  $F_\sigma$  set,  $K_i$  is also an  $F_\sigma$  set in  $X$  as the intersection of two  $F_\sigma$  sets. As  $K_i$  is a subset of  $C_i$ , Proposition 3.16 implies

$$\text{ind}(K_i) \leq n. \tag{9}$$

Because of (9), we can apply Corollary 3.23 to each  $K_i$ : we have

$$K_i = M_i \cup N_i,$$



such that

$$\text{ind}(M_i) \leq n - 1, \quad \text{ind}(N_i) \leq 0.$$

Put  $M := \cup M_i$  and  $N := \cup N_i$ . From (8) we get that

$$X = M \cup N.$$

Each  $M_i$  is an  $F_\sigma$  in  $M$ . For

$$M_i = M_i \cap K_i = M \cap K_i,$$

since  $M_i \subset K_i$ , and  $K_i \cap K_j = \emptyset$ , whenever  $i \neq j$ . Thus  $M_i$  is an  $F_\sigma$  set in  $M$  as the intersection of the set  $K_i$ , which is  $F_\sigma$  in  $X$ , and the subspace  $M$ . Therefore, since Theorem 3.22 was assumed to hold for  $n - 1$ , we can conclude that  $\text{ind}(M) \leq n - 1$ . By a similar argument, every  $N_i$  is an  $F_\sigma$  set in  $N$  and thus  $\text{ind}(N) \leq 0$  by Corollary 3.11.

We thus have  $X = M \cup N$  with  $\text{ind}(M) \leq n - 1$  and  $\text{ind}(N) \leq 0$ . From Proposition 3.19 we conclude, that  $\text{ind}(X) \leq n$ , which completes the proof.  $\square$

**Proposition 3.24.** *If  $C_1$  and  $C_2$  are disjoint closed subsets of a separable metric space  $X$ , and  $A \subset X$  is a subset of dimension  $\leq n$ , where  $n \geq 0$ , there exists a closed separator  $B \subset X$  between the sets  $C_1$  and  $C_2$  such that  $\text{ind}(A \cap B) \leq n - 1$ .*

*Proof.* If  $n = 0$ , then we have two options. The first option is  $\text{ind}(A) = -1$ . Now the proposition holds clearly, since  $X$  is  $T_4$ , and thus there exists a neighbourhood  $V$  of  $C_1$  disjoint from some neighbourhood  $U$  of  $C_2$  and we can choose the boundary  $\partial V$  as the separator. The second option is  $\text{ind}(A) = 0$ , which has already been demonstrated in Proposition 3.9.

Suppose then that  $n > 0$ . Applying Corollary 3.23 we have  $A = D \cup E$ , with  $\text{ind}(D) \leq n - 1$  and  $\text{ind}(E) \leq 0$ . Let us use Proposition 3.9 to obtain a closed separator  $B$  between the sets  $C_1$  and  $C_2$  such that  $B \cap E = \emptyset$ . Thus

$$A \cap B \subset D.$$

Since  $\text{ind}(D) \leq n - 1$ , Proposition 3.16 implies  $\text{ind}(A \cap B) \leq n - 1$ , which completes the proof.  $\square$

**Corollary 3.25.** *If a separable metric space  $X$  has  $\text{ind}(X) \leq n$ , then there exists a closed separator with dimension  $\leq n - 1$  between any two disjoint closed subsets of the space  $X$ .*

*Proof.* This follows from Proposition 3.24 by choosing  $A = X$ .  $\square$

**Proposition 3.26.** *Let  $X$  be a separable metric space with dimension  $\leq n - 1$ , and let  $C_i, C'_i, i = 1, \dots, n$  be  $n$  pairs of closed subsets of  $X$  such that*

$$C_i \cap C'_i = \emptyset.$$

*Then there exist  $n$  closed sets  $B_i$  that act as separators between the sets  $C_i$  and  $C'_i$ , and*

$$\bigcap_{i=1}^n B_i = \emptyset.$$

*Proof.* From Corollary 3.25 we get a closed separator  $B_1$  between the sets  $C_1$  and  $C'_1$  with  $\text{ind}(B_1) \leq n - 2$ . By Proposition 3.24 we get a closed separator  $B_2$  between  $C_2$  and  $C'_2$  with

$$\text{ind}(B_1 \cap B_2) \leq n - 3.$$

By repeatedly applying Proposition 3.24 we arrive at  $k$  sets of closed separators  $B_i$  between the sets  $C_i$  and  $C'_i$  with

$$\text{ind} \left( \bigcap_{i=1}^k B_i \right) \leq n - k - 1.$$

For  $k = n$  we conclude that  $\bigcap_{i=1}^n B_i = \emptyset$ . □

## Brouwer Fixed-Point Theorem

The focus of this chapter is on a very famous result in topology: the Brouwer Fixed-Point Theorem. This theorem allows us to determine the exact dimension of a Euclidean space. We shall give a combinatorial proof of this theorem, which relies on Sperner's lemma. The proof also relies somewhat on the theory of simplexes, the rudiments of which are presented in this chapter. We shall begin by stating a familiar result, which we will need in some proofs.

**Proposition 4.1.** *Suppose  $X$  is a compact topological space, and  $Y$  is a Hausdorff space. Any continuous bijection  $f: X \rightarrow Y$  is a homeomorphism.*

*Proof.* See Väisälä [6, Lause 13.25 and Seuraus 13.26, p. 102].  $\square$

## Some Theory of Simplexes

**Definition 4.2** ( $m$ -simplex, face, vertex, barycentric coordinates). Suppose  $a_0, a_1, \dots, a_m$  are  $m + 1$  linearly independent points of the space  $\mathbb{R}^n$ . The subset of  $\mathbb{R}^n$  consisting of all the points of the form

$$x = \sum_{i=0}^m \lambda_i a_i, \quad (1)$$

where

$$\sum_{i=0}^m \lambda_i = 1, \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for all } i \in \{0, 1, \dots, m\} \quad (2)$$

is called an  $m$ -simplex spanned by the points  $a_0, a_1, \dots, a_m$  and is denoted by  $a_0 a_1 \dots a_m$ . It is evident that an  $m$ -simplex depends only on the points that span it, not on the order of these points.

Consider a simplex  $a_0a_1 \dots a_m \subset \mathbb{R}^n$ . For any  $k+1$  distinct non-negative integers  $j_0, j_1, \dots, j_k$  not larger than  $m$  the points  $a_{j_0}, a_{j_1}, \dots, a_{j_k}$  are linearly independent, so that the  $k$ -simplex  $a_{j_0}a_{j_1} \dots a_{j_k}$  is well defined. Every simplex of the aforementioned form is called a  $k$ -face of the simplex  $a_0a_1 \dots a_m$ . The 0-faces – that is the points  $a_0, a_1$  and so forth – of  $a_0a_1 \dots a_m$  are called *vertices* of  $a_0a_1 \dots a_m$ . The whole simplex  $a_0a_1 \dots a_m$  is also considered to be one of its faces.

The  $k$ -face  $a_{j_0}a_{j_1} \dots a_{j_k}$  consists of all the points of the form (1) satisfying (2) such that

$$\lambda_i = 0 \quad \text{whenever} \quad i \notin \{j_0, j_1, \dots, j_k\}.$$

Since the vertices are linearly independent, every point  $x \in a_0a_1 \dots a_m$  can be represented uniquely in the form (1) under the conditions (2). The coefficients  $\lambda_0, \lambda_1, \dots, \lambda_m$  in (1) are called the *barycentric coordinates* of the point  $x$ . The barycentric coordinates of  $x$  are denoted by  $\lambda_0(x), \lambda_1(x), \dots, \lambda_m(x)$ .

**Proposition 4.3.** *For any  $m+1$  of linearly independent points  $a_0, a_1, \dots, a_m$  of  $\mathbb{R}^n$ , the simplex  $\Delta := a_0a_1 \dots a_m$  is a compact subspace of  $\mathbb{R}^n$  and the barycentric coordinates  $\lambda_0, \lambda_1, \dots, \lambda_m$  are continuous functions from  $\Delta$  to  $[0, 1]$ .*

*Proof.* Let us denote by  $e_i \in \mathbb{R}^{m+1}$ , the unit vector whose  $i^{\text{th}}$  coordinate is 1, and the rest of the coordinates are 0. The points  $e_1, e_2, \dots, e_{m+1}$  are linearly independent, so that the simplex  $S := e_1e_2 \dots e_{m+1} \subset \mathbb{R}^{m+1}$  is well defined. The barycentric coordinates of the points in  $S$  coincide with their coordinates in  $\mathbb{R}^{m+1}$  and by (2)  $S$  is bounded.  $S$  is closed as the intersection of two closed sets, namely the hyperplane  $\{x \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} x_i = 1\}$  and the subset of  $\mathbb{R}^{m+1}$  consisting of the points whose every coordinate is at least 0. Therefore  $S$  is compact as a closed and bounded subset of  $\mathbb{R}^{m+1}$  by Väisälä [6, Laue 13.14, p. 99].

The function  $f: S \rightarrow \Delta$  defined by the rule

$$f(x) = \text{pr}_1(x) \cdot a_0 + \text{pr}_2(x) \cdot a_1 + \dots + \text{pr}_{m+1}(x) \cdot a_m$$

is continuous, since the projection maps are continuous by Väisälä [6, Laue 5.6, p. 43]. The function  $f$  is injective, since the points  $a_0, a_1, \dots, a_m$  are linearly independent. Since  $f(e_i) = a_{i-1}$  for  $i = 1, 2, \dots, m+1$ , we have  $fS = \Delta$ . In other words  $f$  is surjective. Thus  $f$  is a homeomorphism by Proposition 4.1. Hence  $\Delta$  is a compact subspace of  $\mathbb{R}^n$  and the barycentric coordinates  $\lambda_i$ , for  $i = 0, 1, \dots, m$  are continuous as the composition of two continuous functions, since  $\lambda_i(x) = (\text{pr}_{i+1} \circ f^{-1})(x)$ .  $\square$

**Corollary 4.4.** *Any two  $m$ -simplexes are homeomorphic.*

*Proof.* From the proof of Proposition 4.3 we get for any two  $m$ -simplexes  $\Delta_1$  and  $\Delta_2$ , that  $\Delta_1 \approx S \approx \Delta_2$ , from which the claim follows.  $\square$

**Definition 4.5** (Simplicial subdivision, mesh). A *simplicial subdivision* of a simplex  $\Delta \subset \mathbb{R}^n$  is a family  $\mathcal{S} := \{\Delta_i\}_{i=1}^k$  of simplexes in  $\mathbb{R}^n$  satisfying the following three conditions

- The family  $\mathcal{S}$  covers  $\Delta$ , that is,  $\Delta = \bigcup_{i=1}^k \Delta_i$ .
- For any  $i, j \leq k$  the intersection  $\Delta_i \cap \Delta_j$  is either empty or a face of both  $\Delta_i$  and  $\Delta_j$ .
- For  $i = 1, 2, \dots, k$  all faces of  $\Delta_i$  are members of  $\mathcal{S}$ .

The *mesh* of a simplicial subdivision  $\{\Delta_i\}_{i=1}^k$  of a simplex  $\Delta$  is the largest of the numbers  $d(\Delta_1), d(\Delta_2), \dots, d(\Delta_k)$ .

**Definition 4.6** (Barycenter). The *barycenter* of a simplex  $\Delta := a_0 a_1 \dots a_m \subset \mathbb{R}^n$  is the point

$$b(\Delta) = \frac{1}{m+1}a_0 + \frac{1}{m+1}a_1 + \dots + \frac{1}{m+1}a_m.$$

Clearly  $b(\Delta) \in \Delta$  and  $b(\Delta)$  does not belong to any  $k$ -face, where  $k < m$ , of  $\Delta$ .

**Proposition 4.7.** Let  $\Delta := a_0 a_1 \dots a_m \subset \mathbb{R}^n$  be a simplex. For every decreasing sequence  $\Delta_0 \supset \Delta_1 \supset \dots \supset \Delta_k$  of distinct faces of the simplex  $\Delta$ , the points  $b(\Delta_0), b(\Delta_1), \dots, b(\Delta_k)$  are linearly independent. The family  $\mathcal{S}$  of all simplexes of the form  $b(\Delta_0)b(\Delta_1)\dots b(\Delta_k)$  is a simplicial subdivision of the simplex  $\Delta$ . Every  $(m-1)$ -simplex  $S \in \mathcal{S}$  is a face of one or two  $m$ -simplexes of  $\mathcal{S}$ , depending on whether or not the simplex  $S$  is contained in an  $(m-1)$ -face of  $\Delta$ .

*Proof.* Every decreasing sequence of distinct faces of  $\Delta$  can be completed to a sequence  $\Delta_0 \supset \Delta_1 \supset \dots \supset \Delta_m$  consisting of  $m+1$  faces of  $\Delta$  such that

$$\Delta_0 = a_{i_0} a_{i_1} \dots a_{i_m}, \quad \Delta_1 = a_{i_1} a_{i_2} \dots a_{i_m}, \quad \dots, \quad \Delta_m = a_{i_m}, \quad (3)$$

where  $i_0, i_1, \dots, i_m$  is a permutation of  $0, 1, \dots, m$ . To prove the first part of the proposition, it suffices to show that the points  $b(\Delta_0), b(\Delta_1), \dots, b(\Delta_m)$  are linearly independent.

Consider a linear combination

$$\sum_{j=0}^m \mu_j b(\Delta_j). \quad (4)$$

Using the definition of the barycenter, we can represent (4) as a linear combination of the points  $a_{i_0}, a_{i_1}, \dots, a_{i_m}$ . This linear combination is of the form

$$\sum_{j=0}^m \lambda_{i_j} a_{i_j}, \quad (5)$$

where

$$\lambda_{i_j} = \sum_{k=0}^j \frac{1}{m+1-k} \mu_k \quad (6)$$

and

$$\sum_{j=0}^m \lambda_{i_j} = \sum_{j=0}^m \sum_{k=0}^j \frac{1}{m+1-k} \mu_k = \sum_{i=0}^m \sum_{j=i}^m \frac{1}{m+1-i} \mu_i = \sum_{i=0}^m \mu_i. \quad (7)$$

Suppose the linear combination (4) is equal to 0. As the points  $a_{i_0}, a_{i_1}, \dots, a_{i_m}$  are linearly independent, from (5) we get that  $\lambda_{i_j} = 0$  for all  $j \in \{0, 1, \dots, m\}$ . Now from (6) it follows that  $\mu_k = 0$  for all  $k \in \{0, 1, \dots, m\}$ . Thus the points  $b(\Delta_0), b(\Delta_1), \dots, b(\Delta_m)$  are linearly independent and the simplex spanned by the barycenters is well-defined.

Every simplex in  $\mathcal{S}$  is a face of an  $m$ -simplex of the form  $b(\Delta_0)b(\Delta_1) \dots b(\Delta_m)$ . From (4)–(7) it follows that the simplex  $b(\Delta_0)b(\Delta_1) \dots b(\Delta_m)$  is a subset of  $\Delta$ , when the coefficients  $\mu_j$  in (4) are chosen in such a way that they satisfy the conditions in (2). We shall show that  $b(\Delta_0)b(\Delta_1) \dots b(\Delta_m)$  coincides with the set

$$F := \{x \in \Delta : \lambda_{i_0}(x) \leq \lambda_{i_1}(x) \leq \dots \leq \lambda_{i_m}(x)\}. \quad (8)$$

By (6), it suffices to show that every point  $x \in F$  can be represented in the form (4) with  $\sum_{j=0}^m \mu_j = 1$  and  $\mu_j \geq 0$  for  $j \in \{0, 1, \dots, m\}$ . We get such a representation by choosing

$$\mu_0(x) = (m+1)\lambda_{i_0}(x) \quad \text{and} \quad \mu_j(x) = (m+1-j)(\lambda_{i_j}(x) - \lambda_{i_{j-1}}(x)) \quad (9)$$

for  $j = 1, 2, \dots, m$ , since  $\sum_{j=0}^m \mu_j(x) = \sum_{j=0}^m \lambda_{i_j}(x) = 1$  and clearly every  $\mu_j(x)$  is non-negative.

From (9) it follows that the faces of the simplex  $b(\Delta_0)b(\Delta_1) \dots b(\Delta_m)$  can be described by adding to condition (8) a number of conditions of the form  $\lambda_{i_j}(x) = \lambda_{i_{j-1}}(x)$ , where  $1 \leq j \leq m$  and possibly the condition  $\lambda_{i_0}(x) = 0$ . As the intersection of a face determined by such conditions with a face of another simplex  $b(\Delta'_0)b(\Delta'_1) \dots b(\Delta'_m)$  corresponding to a different permutation of  $0, 1, \dots, m$  still satisfies the aforementioned conditions, or is empty, the family  $\mathcal{S}$  satisfies the second condition of Definition 4.5. The first condition of Definition 4.5 is also satisfied, since any point in  $\Delta$  belongs to a set of the

form (8). By the definition of  $\mathcal{S}$ , every face of a simplex in  $\mathcal{S}$  belongs to  $\mathcal{S}$ , so that the third condition of Definition 4.5 holds. Therefore  $\mathcal{S}$  is a simplicial subdivision of  $\Delta$ .

Let  $S := b(\Delta_0)b(\Delta_1) \dots b(\Delta_{m-1})$  be an  $(m-1)$ -simplex in  $\mathcal{S}$ . If the simplex  $S$  is contained in an  $(m-1)$ -face of  $\Delta$ , in other words, if  $\Delta_0 \neq \Delta$ , then  $S$  is a face of exactly one  $m$ -simplex of  $\mathcal{S}$ , namely  $b(\Delta)b(\Delta_0)b(\Delta_1) \dots b(\Delta_{m-1})$ . If, on the other hand  $S$  is not contained in any  $(m-1)$ -face of  $\Delta$ , i.e.  $\Delta_0 = \Delta$ , then either  $\Delta_{m-1}$  is a 2-simplex, or there exists a  $j < m-1$  such that the simplex  $\Delta_j$  is obtained from  $\Delta_{j-1}$  by removing two vertices, in which case  $\Delta_{m-1}$  is a 1-simplex. In the first case  $S$  is the face of exactly two  $m$ -simplexes, namely  $b(\Delta_0)b(\Delta_1) \dots b(\Delta_{m-1})b(a)$  and  $b(\Delta_0)b(\Delta_1) \dots b(\Delta_{m-1})b(a')$  where  $a$  and  $a'$  are the two vertices of  $\Delta_{m-1}$ . In the second case  $S$  is the face of the two  $m$ -simplexes that we get when we remove only one vertice from  $\Delta_{j-1}$  to get  $\Delta_j$ .  $\square$

**Definition 4.8** (Barycentric subdivision). The simplicial subdivision defined in Proposition 4.7 is called the *barycentric subdivision* of  $\Delta$ . This is the first barycentric subdivision of  $\Delta$ . If the  $j^{\text{th}}$  barycentric subdivision  $\{\Delta_i\}_{i=1}^k$  of  $\Delta$  is already defined, the  $(j+1)^{\text{th}}$  barycentric subdivision is defined as the union  $\bigcup_{i=1}^k \mathcal{S}_i$ , where  $\mathcal{S}_i$  is the barycentric subdivision of  $\Delta_i$ . This union is a simplicial subdivision of  $\Delta$ .

**Lemma 4.9.** Suppose  $\Delta = a_0a_1 \dots a_m \subset \mathbb{R}^n$  is an  $m$ -simplex and let  $x \in \Delta$ , i.e.  $x = \sum_{j=0}^m \lambda_j a_j$ , where  $\sum_{j=0}^m \lambda_j = 1$  and  $\lambda_j \geq 0$ . For every point  $y \in \mathbb{R}^n$  we have the inequality

$$d(x, y) \leq \max_{j \leq m} d(a_j, y).$$

*Proof.* We have

$$\begin{aligned} d(x, y) &= \|x - y\| = \left\| \sum_{j=0}^m \lambda_j a_j - \sum_{j=0}^m \lambda_j y \right\| = \left\| \sum_{j=0}^m \lambda_j (a_j - y) \right\| \\ &\leq \sum_{j=0}^m \lambda_j \|a_j - y\| \leq \max_{j \leq m} \|a_j - y\| \sum_{j=0}^m \lambda_j \\ &= \max_{j \leq m} \|a_j - y\| = \max_{j \leq m} d(a_j, y). \end{aligned}$$

$\square$

**Lemma 4.10.** The diameter of a simplex  $a_0a_1 \dots a_m \subset \mathbb{R}^n$  is equal to the diameter of the set  $\{a_0, a_1, \dots, a_m\}$ .

*Proof.* Let  $x, y \in a_0a_1 \dots a_m$ . From Lemma 4.9 we get that

$$d(x, y) \leq \max_{j \leq m} d(a_j, y).$$

Applying Lemma 4.9 again, we get

$$d(a_j, y) \leq \max_{i \leq m} d(a_j, a_i).$$

Thus  $d(x, y) \leq \max_{i, j \leq m} d(a_i, a_j)$ , which proves the claim.  $\square$

**Lemma 4.11.** *The mesh of the barycentric subdivision of the  $m$ -simplex  $\Delta := a_0 a_1 \dots a_m$  is at most  $(m/(m+1))d(\Delta)$ .*

*Proof.* By virtue of Lemma 4.10 it suffices to show that the distance between any two points of the form

$$b(\Delta_j) = \frac{1}{j+1}(a_{i_0} + a_{i_1} + \dots + a_{i_j}) \quad \text{and} \quad b(\Delta_k) = \frac{1}{k+1}(a_{i_0} + a_{i_1} + \dots + a_{i_k}),$$

where  $k < j \leq m$ , and  $i_0, i_1, \dots, i_m$  is some permutation of  $0, 1, \dots, m$  is at most  $(m/(m+1))d(\Delta)$ . From Lemma 4.9 we get that

$$d(b(\Delta_k), b(\Delta_j)) \leq d(a_{i_l}, b(\Delta_j))$$

for some  $0 \leq l \leq m$ . Thus

$$\begin{aligned} d(b(\Delta_k), b(\Delta_j)) &\leq d(a_{i_l}, b(\Delta_j)) \\ &= \|b(\Delta_j) - a_{i_l}\| \\ &= \left\| \frac{1}{j+1}(a_{i_0} + a_{i_1} + \dots + a_{i_j}) - a_{i_l} \right\| \\ &= \frac{1}{j+1} \left\| \sum_{h=0}^j (a_{i_h} - a_{i_l}) \right\| \\ &\leq \frac{1}{j+1} \sum_{h=0}^j \|a_{i_h} - a_{i_l}\| \\ &\leq \frac{j}{j+1} d(\Delta) \\ &\leq \frac{m}{m+1} d(\Delta). \end{aligned}$$

$\square$

**Corollary 4.12.** *For every simplex  $\Delta$  and for every  $\varepsilon > 0$  there exists a natural number  $l$  such that the mesh of the  $l^{\text{th}}$  barycentric subdivision is less than  $\varepsilon$ .*



*Proof.* By Lemma 4.11 the mesh of the second barycentric subdivision of the  $m$ -simplex is at most

$$\left(\frac{m}{m+1}\right)^2 d(\Delta)$$

and similarly the mesh of the  $l^{\text{th}}$  barycentric subdivision is at most

$$\left(\frac{m}{m+1}\right)^l d(\Delta).$$

If  $\varepsilon > 0$ , by choosing  $l$  large enough, we get that

$$\left(\frac{m}{m+1}\right)^l d(\Delta) < \varepsilon.$$

□

**Proposition 4.13** (Sperner's lemma). *Let  $\mathcal{S}$  be the  $l^{\text{th}}$  barycentric subdivision of an  $m$ -simplex  $a_0 a_1 \dots a_m$  and let  $V$  be the set of all vertices of simplexes in  $\mathcal{S}$ . If a function  $h: V \rightarrow \{0, 1, \dots, m\}$  satisfies the condition*

$$h(v) \in \{i_0, i_1, \dots, i_k\} \quad \text{whenever} \quad v \in a_{i_0} a_{i_1} \dots a_{i_k},$$

*then the number of simplexes in  $\mathcal{S}$ , on the vertices of which  $h$  assumes all values from 0 to  $m$ , is odd.*

*Proof.* We apply induction with respect to  $m$ . Sperner's lemma holds for  $m = 0$ , since  $\mathcal{S} = \{a_0\}$ , and  $h(a_0) = 0$ . Suppose the lemma holds for some  $m = n - 1$ . Consider an  $n$ -simplex  $\Delta := a_0 a_1 \dots a_n$ , the  $l^{\text{th}}$  barycentric subdivision  $\mathcal{S}$  of  $\Delta$  and a function  $h$  satisfying the condition in Sperner's lemma.

We shall consider the number of  $(n - 1)$ -simplexes in  $\mathcal{S}$ , the vertices of which map onto  $\{0, 1, \dots, n - 1\}$  under  $h$ . Denote the set consisting of these  $(n - 1)$ -simplexes by  $S$ .

Let  $r$  be the number of  $n$ -simplexes in  $\mathcal{S}$ , whose vertices receive all the values from 0 to  $n$  under  $h$ . Each of these simplexes contributes exactly one  $(n - 1)$ -face to the set  $S$ . Let  $q$  denote the number of  $n$ -simplexes in  $\mathcal{S}$ , the vertices of which map onto  $\{0, 1, \dots, n - 1\}$  under  $h$ . That is to say, two vertices map to the same value under  $h$ . These simplexes contribute two  $(n - 1)$ -faces each to the set  $S$ . We arrive at the figure  $r + 2q$ . Note that this number is greater than  $\#S$ , since we have counted some of the  $(n - 1)$ -faces twice; some of these faces are faces of two  $n$ -simplexes.

Let  $x$  denote the number of elements of  $S$ , which lie on an  $(n - 1)$ -face of  $\Delta$ . It follows from the definition of the function  $h$ , that the only face of  $\Delta$  on

which the elements of  $S$  can lie, is the face  $a_0a_1 \dots a_{n-1}$ . By the induction hypothesis,  $x$  is an odd number. Let  $y$  be the number of the rest of the elements of  $S$ , i.e.  $y := \#S - x$ .

From the last part of Proposition 4.7, it follows that every  $(n-1)$ -simplex is a face of one or two  $n$ -simplexes: one if the simplex is contained in an  $(n-1)$ -face of  $\Delta$  and two if this is not the case. Thus in the figure  $r + 2q$  we have counted the simplexes that contribute to the number  $y$  twice, and the simplexes that contribute to the number  $x$  only once. Thus, counting in two different ways, we arrive at the equation

$$r + 2q = x + 2y,$$

whence it follows that  $r$  is an odd number, since  $x$  is odd. This completes the proof.  $\square$

The next result was first proved by Knaster, Kuratowski and Mazurkiewicz. This is why it is known as the KKM lemma.

**Proposition 4.14** (The KKM lemma). *Let  $\{F_i\}_{i=0}^m$  be a family of closed subsets of a simplex  $\Delta := a_0a_1 \dots a_m$ . If for every face  $a_{i_0}a_{i_1} \dots a_{i_k}$  of  $\Delta$  it holds that*

$$a_{i_0}a_{i_1} \dots a_{i_k} \subset \bigcup_{r=0}^k F_{i_r},$$

*then the intersection*

$$\bigcap_{i=0}^m F_i$$

*is non-empty.*

*Proof.* Suppose, for a contradiction, that  $\bigcap_{i=0}^m F_i = \emptyset$ . The collection  $\{U_i\}_{i=0}^k$ , where  $U_i := \Delta \setminus F_i$  is an open covering of  $\Delta$ . Since  $\Delta$  is compact, by Proposition 2.33 there exists an  $\varepsilon > 0$  such that every subset of  $\Delta$ , whose diameter is less than  $\varepsilon$  is contained in a set  $U_i$ , i.e. disjoint from  $F_i$ .

By Corollary 4.12 we can choose the natural number  $l$  to be such, that the mesh of the  $l^{\text{th}}$  barycentric subdivision  $\mathcal{S}$  of  $\Delta$  is less than  $\varepsilon$ . Denote by  $V$  the set of vertices of simplexes in  $\mathcal{S}$ . We construct a function  $h: V \rightarrow \{0, 1, \dots, m\}$  as follows. For every  $v \in V$  the intersection of all faces of  $\Delta$  that contain  $v$  is a face of  $\Delta$ . In other words, this intersection is of the form  $a_{i_0}a_{i_1} \dots a_{i_k}$ . Since  $v \in a_{i_0}a_{i_1} \dots a_{i_k}$ , by the assumption of the proposition there exists a number  $j \leq k$  such that  $v \in F_{i_j}$ . We set  $h(v) = i_j$ . By construction  $h$  satisfies the assumptions of Sperner's lemma, so that there exists at least one simplex  $S := v_0v_1 \dots v_m \in \mathcal{S}$  such that  $h(v_i) = i$ , for

$i = 0, 1, \dots, m$ . Thus for  $i = 0, 1, \dots, m$  we have  $v_i \in F_i$ , which in turn implies that  $S \cap F_i \neq \emptyset$  for  $i = 0, 1, \dots, m$ . This is a contradiction, since the diameter of  $S$  is less than  $\varepsilon$ .  $\square$

**Lemma 4.15.** *Every compact convex subset  $A \subset \mathbb{R}^n$ , with  $\text{int}A \neq \emptyset$  is homeomorphic to the closed unit ball  $\overline{B}^n$ , and the boundary  $\partial A$  is a homeomorph of the unit sphere  $S^{n-1}$ .*

*Proof.* Without loss of generality, we may assume that  $\bar{0} \in \text{int}A$ . Every ray that emanates from the origin intersects with the boundary  $\partial A$  exactly once: should some ray intersect the boundary more than once,  $A$  would not be convex, and should the intersection be empty,  $A$  would not be bounded and thus would not be compact. Hence every point of  $A \setminus \{\bar{0}\}$  lies on one, and only one ray emanating from the origin.

The map  $f: \partial A \rightarrow S^{n-1}$ , defined with the rule

$$f(x) = \frac{x}{\|x\|}$$

is a continuous bijection from a compact space to a Hausdorff space and is thus a homeomorphism by Proposition 4.1. Hence  $f: \partial A \approx S^{n-1}$ .

Let  $g: \overline{B}^n \rightarrow A$  be a function defined by the rule

$$g(x) = \begin{cases} \|x\| f^{-1}\left(\frac{x}{\|x\|}\right), & \text{for } x \neq \bar{0} \\ 0, & \text{for } x = \bar{0} \end{cases}.$$

The function  $g$  is continuous except maybe at the origin. Since  $A$  is compact the norms of the elements of  $A$  are bounded. Put  $M := \sup\{\|x\| : x \in A\}$ . Now for all  $x \in \overline{B}^n$  we have  $\|g(x)\| \leq M \|x\|$ . To prove that  $g$  is continuous at the origin, let  $\varepsilon > 0$ . Now  $\|g(x)\| < \varepsilon$ , whenever  $\|x\| < \delta = \varepsilon/M$ . Thus  $g$  is a continuous function. Since  $g$  is bijective, it is a homeomorphism by Proposition 4.1. This proves the claim.  $\square$

**Theorem 4.16** (Brouwer Fixed-Point Theorem). *If  $f: \overline{B}^n \rightarrow \overline{B}^n$  is a continuous function, there exists a point  $x \in \overline{B}^n$  such that  $f(x) = x$ .*

*Proof.* By virtue of Lemma 4.15 we can replace the ball  $\overline{B}^n$  by an  $n$ -simplex  $\Delta := a_0 a_1 \dots a_n$ , with a non-empty interior and investigate the behaviour of a continuous function  $f: \Delta \rightarrow \Delta$ .

For every point  $x \in \Delta$ , we have

$$x = \lambda_0(x)a_0 + \lambda_1(x)a_1 + \dots + \lambda_n(x)a_n, \quad (10)$$

where

$$\sum_{i=0}^n \lambda_i(x) = 1 \quad (11)$$

and  $\lambda_i(x) \geq 0$  for  $i = 0, 1, \dots, n$ .

The image of the point  $x \in \Delta$  under the function  $f$  can be written as

$$f(x) = \lambda_0(f(x))a_0 + \lambda_1(f(x))a_1 + \dots + \lambda_n(f(x))a_n, \quad (12)$$

where

$$\sum_{i=0}^n \lambda_i(f(x)) = 1 \quad (13)$$

and  $\lambda_i(f(x)) \geq 0$  for  $i = 0, 1, \dots, n$ .

For  $i = 0, 1, \dots, n$  the set

$$F_i := \{x \in \Delta : \lambda_i(f(x)) \leq \lambda_i(x)\} \quad (14)$$

is closed as the preimage of a closed set under a continuous function. We shall next show, that the collection  $\{F_i\}_{i=0}^n$  satisfies the assumptions of Proposition 4.14. Let  $a_{i_0}a_{i_1}\dots a_{i_k}$  be a face of  $\Delta$ . Consider a point  $x \in a_{i_0}a_{i_1}\dots a_{i_k}$ . We have

$$\sum_{j=0}^k \lambda_{i_j}(x) = 1,$$

so that by (13) we get

$$\sum_{j=0}^k \lambda_{i_j}(f(x)) \leq \sum_{j=0}^k \lambda_{i_j}(x).$$

Hence  $\lambda_{i_j}(f(x)) \leq \lambda_{i_j}(x)$  for at least one  $j \leq k$ , whence it follows that  $x \in F_{i_j}$ . Hence we have shown, that

$$a_{i_0}a_{i_1}\dots a_{i_k} \subset \bigcup_{j=0}^k F_{i_j}.$$

Now, by Proposition 4.14 there exists a point  $x \in \bigcap_{i=0}^n F_i$ . From (14) it follows that  $\lambda_i(f(x)) \leq \lambda_i(x)$  for  $i = 0, 1, \dots, n$ . However, the strict inequality cannot hold for any  $i$  because of (11) and (13). Therefore,  $\lambda_i(f(x)) = \lambda_i(x)$  for  $i = 0, 1, \dots, n$ , whence it follows that  $f(x) = x$ , because of (10) and (12). This proves the claim.  $\square$

**Corollary 4.17.** *There exists no continuous function  $r: \overline{B^n} \rightarrow S^{n-1}$  which keeps each point of the boundary  $S^{n-1}$  fixed.*

*Proof.* Suppose a continuous function  $r: \overline{B^n} \rightarrow S^{n-1}$  which keeps the points on the boundary fixed exists. Let  $f: S^{n-1} \rightarrow \overline{B^n}$  be the function which maps each point  $x \in S^{n-1}$  to its antipode, i.e.  $f(x) = -x$ . Now the composition  $f \circ r: \overline{B^n} \rightarrow \overline{B^n}$  is continuous, and contains no fixed-point, which contradicts the Brouwer Fixed-Point Theorem.  $\square$

**Proposition 4.18.** *Consider the cube  $I^n := [-1, 1]^n \subset \mathbb{R}^n$ . Let  $F_i^+$  be the face of the cube determined by the equation  $x_i = 1$ , where  $x_i$  denotes the  $i^{\text{th}}$  coordinate of a point of  $I^n$ , and let  $F_i^-$  be the opposite face, i.e.  $x_i = -1$ .*

*Let  $C_i$  be a closed separator between  $F_i^+$  and  $F_i^-$ . Then*

$$\bigcap_{i=1}^n C_i \neq \emptyset.$$

*Proof.* Suppose  $C_i$  is a closed separator between  $F_i^+$  and  $F_i^-$ . Hence there exists sets  $U_i^+, U_i^- \subset I^n$  such that

$$I^n \setminus C_i = U_i^+ \cup U_i^-$$

$$F_i^+ \subset U_i^+, \quad F_i^- \subset U_i^-$$

and

$$U_i^+ \cap U_i^- = \emptyset,$$

with both  $U_i^+$  and  $U_i^-$  open in  $I^n \setminus C_i$  and thus open in  $I^n$ , since  $I^n \setminus C_i$  is open in  $I^n$ . For each point  $x \in I^n$  let  $v_x$  be the point whose  $i^{\text{th}}$  component has the value

$$\pm d(x, C_i),$$

where the sign is positive, if  $x \in U_i^+$ , and negative, if  $x \in U_i^-$ . We define a function  $f: I^n \rightarrow I^n$  with the rule

$$f(x) = x + v_x.$$

This function is well-defined, since the way the sign of the coordinates of the vector  $v_x$  is determined, the image of  $x$  under  $f$  is in  $I^n$ . The function  $f$  is continuous, since the distance function is continuous. We can thus apply the Brouwer Fixed-Point Theorem (Theorem 4.16), for  $\overline{B^n}$  and  $I^n$  are homeomorphic. Thus there exists a point  $x_0$  such that

$$f(x_0) = x_0.$$

This means that  $d(x_0, C_i) = 0$  for all  $i \in \{1, 2, \dots, n\}$ , and since each  $C_i$  is closed, we have  $x_0 \in C_i$ . Thus the intersection of the sets  $C_i$ , where  $i \in \{1, 2, \dots, n\}$ , is non-empty. This completes the proof.  $\square$

## Some Properties of $\mathbb{R}^n$

We have not yet demonstrated that sets of dimension  $> 0$  even exist. We shall fix this shortcoming in this section. We shall prove a very important result, namely that the dimension of the Euclidean space  $\mathbb{R}^n$  is exactly  $n$ . Our theory of dimension would violate intuition quite seriously, should this not be the case.

**Theorem 4.19.** *The real line  $\mathbb{R}$  has  $\text{ind}(\mathbb{R}) = 1$ .*

*Proof.* Since  $\mathbb{R}$  is connected, no real number has arbitrarily small neighbourhoods, whose boundaries are empty. Hence we get  $\text{ind}(\mathbb{R}) \geq 1$ .

Suppose  $x \in \mathbb{R}$  and let  $U$  be a neighbourhood of  $x$ . Now  $U$  contains some open interval  $]\alpha, \beta[ \ni x$ . The topology on the subspace  $\partial]\alpha, \beta[ = \{\alpha, \beta\}$  is the subspace topology, which is discrete. Now the open and closed neighbourhood  $\{\alpha\}$  of the point  $\alpha$  is contained in every neighbourhood of  $\alpha$ . The analogous is true for the other point  $\beta$ . Thus  $\text{ind}(\partial]\alpha, \beta[) = 0$ . Now it follows from (SID2) of Definition 3.13 that  $\text{ind}(\mathbb{R}) \leq 1$ , which proves the claim.  $\square$

**Corollary 4.20.** *Any non-empty interval  $J \subset \mathbb{R}$  has  $\text{ind}(\mathbb{R}) = 1$ .*

*Proof.* The interval  $J$  is connected, thus no element of the interval has arbitrarily small neighbourhoods with empty boundaries. Hence  $\text{ind}(J) \geq 1$ . As  $J \subset \mathbb{R}$ , Proposition 3.16 and Theorem 4.19 give us  $\text{ind}(J) \leq 1$ .  $\square$

By a similar argument we can show that the sphere  $S^1$  also has  $\text{ind}(S^1) = 1$ . Every point in the sphere has arbitrarily small neighbourhoods, whose boundaries are two-point sets. Hence  $\text{ind}(S^1) \leq 1$ . Since  $S^1$  is connected, no point in  $S^1$  has arbitrarily small open and closed neighbourhoods, whence it follows that  $\text{ind}(S^1) > 0$ .

**Proposition 4.21.** *The spaces  $\mathbb{R}^n$  and  $S^n$  are at most  $n$ -dimensional.*

*Proof.* The base case  $n = 1$  follows from Theorem 4.19 and the discussion below it. Suppose the proposition holds for spaces whose dimensions do not exceed some  $n \geq 1$ . For every point  $x$  in the Euclidean space  $\mathbb{R}^{n+1}$ , or the sphere  $S^{n+1}$  and for each neighbourhood  $V$  of the point  $x$  there exists a neighbourhood  $U \subset V$  of  $x$ , whose boundary is homeomorphic to  $S^n$ . Hence, by Theorem 3.15 and the induction hypothesis, we have  $\text{ind}(\partial U) \leq n$ , which implies  $\text{ind}(\mathbb{R}^{n+1}) \leq n + 1$ , and  $\text{ind}(S^{n+1}) \leq n + 1$ .  $\square$

**Corollary 4.22.** *The  $n$ -cube  $I^n := [-1, 1]^n$  has  $\text{ind}(I^n) \leq n$ .*

*Proof.* Since  $I^n \subset \mathbb{R}^n$ , the claim follows from Propositions 3.16 and 4.21.  $\square$

**Proposition 4.23.**  $\text{ind}(I^n) \geq n$ .

*Proof.* Suppose, that  $\text{ind}(I^n) \leq n - 1$ . Then by Proposition 3.26 there exists  $n$  closed subsets  $B_i \subset I^n$ , for  $i = 1, 2, \dots, n$ , with each  $B_i$  being a separator between two opposite faces of  $I^n$ , and  $\bigcap_{i=1}^n B_i = \emptyset$ . This contradicts Proposition 4.18.  $\square$

**Corollary 4.24.**  $\text{ind}(I^n) = n$ .

*Proof.* This follows from Corollary 4.22 and Proposition 4.23.  $\square$

**Proposition 4.25.**  $\text{ind}(\mathbb{R}^n) \geq n$ .

*Proof.* Since by Proposition 4.23  $\text{ind}(I^n) \geq n$ , and since  $I^n \subset \mathbb{R}^n$ , the claim follows from Proposition 3.16.  $\square$

**Corollary 4.26.** *The sphere  $S^n \subset \mathbb{R}^{n+1}$  has  $\text{ind}(S^n) = n$ .*

*Proof.* Any point  $x \in \mathbb{R}^{n+1}$  has arbitrarily small neighbourhoods, whose boundaries are homeomorphic to  $S^n$ . By Proposition 4.25 we have  $\text{ind}(\mathbb{R}^{n+1}) \geq n + 1$ , whence it follows that  $\text{ind}(S^n) \geq n$ , which together with Proposition 4.21 proves the claim.  $\square$

The above corollary explains the notation  $S^{n-1}$  for the boundary of a ball  $B^n$ . The dimension of the boundary is strictly smaller than the dimension of the ball itself. Now we are finally ready to determine the exact dimension of a Euclidean space.

**Theorem 4.27.** *The Euclidean space  $\mathbb{R}^n$  has dimension  $n$ .*

*Proof.* This follows from Propositions 4.21 and 4.25.  $\square$

**Theorem 4.28.** *The spaces  $\mathbb{R}^n$  and  $\mathbb{R}^m$  are homeomorphic, if and only if  $n = m$ .*

*Proof.* It is evident, that  $\mathbb{R}^n \approx \mathbb{R}^m$ , if  $n = m$ . The converse follows from Theorems 3.15 and 4.27.  $\square$





## Invariance of Domain

We are almost sufficiently prepared to prove Brouwer's Theorem on the Invariance of Domain. The proof of this theorem, which we shall present shortly, relies on results concerning the extensions of continuous functions. Thus most of this chapter is devoted to developing this theory. Having these new tools at hand, we conclude this chapter, and hence the whole thesis, by proving the Invariance of Domain.

**Definition 5.1** (Stable and unstable values). Suppose  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces and  $f: X \rightarrow Y$  a continuous function. A point  $y \in fX$  is called an *unstable value* of the function  $f$ , if for every  $\varepsilon > 0$  there exists a continuous function  $g: X \rightarrow Y$  satisfying

$$d_Y(f(x), g(x)) < \varepsilon, \quad \text{for every } x \in X,$$

$$gX \subset Y \setminus \{y\}.$$

If a point of  $fX$  is not an unstable value of  $f$ , it is a *stable value*.

**Example 5.2.** Suppose  $X$  is a topological space and  $f: X \rightarrow I^n$  a continuous function. Now every point on the boundary of  $I^n$  is unstable. For given any  $0 < \varepsilon < 1$ , the functions

$$g_i(x) = (1 - \varepsilon)f_i(x), \quad i = 1, 2, \dots, n$$

define a continuous function whose image does not contain any boundary points of  $I^n$ .

**Proposition 5.3.** Suppose  $(X, d)$  is a separable metric space with  $\text{ind}(X) < n$ , and that  $f: X \rightarrow I^n$  is a continuous map. Now all values of  $f$  are unstable.

*Proof.* Example 5.2 shows that it suffices to show, that there are no stable interior points of  $I^n$ . To prove this, it is enough to show that the origin is not a stable value of  $f$ . We consider the coordinate functions  $f_i$ , where  $i = 1, 2, \dots, n$ , of  $f$ . Suppose  $0 < \varepsilon < 1$ . Let  $C_i^+$  be the subset of  $X$ , for the points of which it holds that

$$f_i(x) \geq \varepsilon,$$

and let  $C_i^-$  be the set of points of  $X$  for which

$$f_i(x) \leq -\varepsilon.$$

For each  $i$ , the sets  $C_i^+$  and  $C_i^-$  are closed and disjoint. Hence by Proposition 3.26 there exist closed sets  $B_1, B_2, \dots, B_n$  such that  $B_i$  is a separator between the sets  $C_i^+$  and  $C_i^-$ . In other words, there exist disjoint open sets  $U_i^+ \supset C_i^+$  and  $U_i^- \supset C_i^-$  such that

$$X \setminus B_i = U_i^+ \cup U_i^-,$$

and

$$\bigcap_{i=1}^n B_i = \emptyset. \quad (1)$$

We shall define functions  $g_1, g_2, \dots, g_n: X \rightarrow [-1, 1]$  with the following rules:

$$\begin{aligned} g_i(x) &= f_i(x), & \text{if } x \in C_i^+ \cup C_i^- \\ g_i(x) &= \varepsilon \frac{d(x, B_i)}{d(x, C_i^+) + d(x, B_i)}, & \text{if } x \in \text{cl}_X(U_i^+ \setminus C_i^+) \\ g_i(x) &= -\varepsilon \frac{d(x, B_i)}{d(x, C_i^-) + d(x, B_i)}, & \text{if } x \in \text{cl}_X(U_i^- \setminus C_i^-) \\ g_i(x) &= 0, & \text{if } x \in B_i. \end{aligned}$$

We shall show that the functions  $g_i$  are continuous. It is clear that the subfunctions that form  $g_i$  are continuous. Since all of the domains of the subfunctions are closed in  $X$ , if the subfunctions whose domains intersect attain the same values in this intersection, then the whole piecewise defined function  $g_i$  is continuous.

In the intersection  $(C_i^+ \cup C_i^-) \cap \text{cl}_X(U_i^+ \setminus C_i^+) = \partial C_i^+$ , we have  $f_i(x) = \varepsilon$ , and

$$\varepsilon \frac{d(x, B_i)}{d(x, C_i^+) + d(x, B_i)} = \varepsilon \frac{d(x, B_i)}{0 + d(x, B_i)} = \varepsilon.$$

Similarly in  $(C_i^+ \cup C_i^-) \cap \text{cl}_X(U_i^- \setminus C_i^-) = \partial C_i^-$ , we have  $f_i(x) = -\varepsilon$ , and

$$-\varepsilon \frac{d(x, B_i)}{d(x, C_i^-) + d(x, B_i)} = -\varepsilon \frac{d(x, B_i)}{0 + d(x, B_i)} = -\varepsilon.$$

The subfunctions, whose domain has a non-empty intersection with  $B_i$  vanish in this intersection, as can be seen from the definitions of the subfunctions. If the sets  $\text{cl}_X(U_i^+ \setminus C_i^+)$  and  $\text{cl}_X(U_i^- \setminus C_i^-)$  meet, they meet in  $B_i$ , and thus the relevant subfunctions attain the value zero in this intersection.

We have checked all non-empty intersections. Thus the functions  $g_i$  are continuous, and we have

$$|g_i(x) - f_i(x)| \leq 2\varepsilon \quad (2)$$

for all  $x \in X$ . By (1) there is no point in  $X$  such that all the functions  $g_i$  vanish simultaneously, since  $g_i(x) = 0$  only when  $x \in B_i$ . Hence the origin is not an image point of the continuous function  $g := (g_1, g_2, \dots, g_n)$ , and with (2) this shows that the origin is not a stable value of  $f$ .  $\square$

**Proposition 5.4.** *Suppose  $X$  is a separable metric space and suppose  $f: X \rightarrow I^n$  is continuous. If there exists a point  $y \in I^n$  such that*

$$fX \subset I^n \setminus \{y\},$$

*then for every  $\varepsilon > 0$  there exists a continuous function  $g: X \rightarrow I^n$  such that*

$$\|f(x) - g(x)\| < \varepsilon \quad \text{for every } x \in X$$

$$\overline{gX} \subset I^n \setminus \{y\}.$$

*Proof.* If the point  $y$  lies on the boundary of  $I^n$ , the function defined in Example 5.2 satisfies the desired conditions. Suppose that  $y$  is an interior point of  $I^n$ . Let  $f'(x)$  be the projection of the point  $f(x)$  from  $y$  on the boundary  $\partial I^n$ . If the length of the segment joining  $f(x)$  and  $f'(x)$  is greater than, or equal to  $\varepsilon/2$ , we define the point  $g(x)$  to be the point on this segment at distance  $\varepsilon/2$  from  $f(x)$ . If the length of this line segment is less than  $\varepsilon/2$ , we put  $g(x) = f'(x)$ . The function  $g: X \rightarrow I^n$  is as wanted.  $\square$

**Proposition 5.5.** *Suppose  $(Y, d)$  is a compact metric space and let  $X$  be any space. Then the space  $(C(X, Y), \|\cdot\|_\infty)$  \* is complete.*

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\*Here we mean the space consisting of all the continuous functions from  $X$  to  $Y$  with the sup-metric which is naturally defined as  $\|f - g\|_\infty := \sup_{x \in X} \{d(f(x), g(x))\}$ .

*Proof.* Let  $(f_n)$  be a Cauchy sequence in  $C(X, Y)$ . Now for each  $x \in X$  the sequence  $(f_n(x))$  is Cauchy in  $Y$ , since  $d(f_m(x), f_n(x)) \leq \|f_n - f_m\|_\infty$ . Since  $Y$  is a compact metric space and therefore complete by Väisälä [6, Lause 13.28, p. 102], the sequence  $(f_n(x))$  converges to a point  $f(x)$ . The sequence  $(f_n)$  converges uniformly to  $f$ , and thus  $f \in C(X, Y)$  by Väisälä [7, Lause 10.13, p. 81].  $\square$

Proposition 5.5 allows us to apply the Baire Category Theorem to the space  $(C(X, Y), \|\cdot\|_\infty)$ , where  $Y$  is a compact metric space. To refresh our memory, we shall state the theorem here.

**Proposition 5.6** (Baire Category Theorem). *Suppose  $(X, d)$  is a complete metric space, and suppose  $(G_j)_{j \in \mathbb{N}}$  is a sequence of open dense subsets of  $X$ . Then the intersection  $\bigcap_{i \in \mathbb{N}} G_i$  is dense in  $X$ .*

*Proof.* Väisälä [7, Lause 10.8, p. 78]  $\square$

**Corollary 5.7.** *The countable intersection of dense  $G_\delta$  sets in a complete metric space is a dense  $G_\delta$  set.*

*Proof.* Suppose  $(X, d)$  is a complete metric space, and suppose  $(G_j)_{j \in \mathbb{N}}$  is a sequence of dense  $G_\delta$  subsets of  $X$ . Each  $G_j$  is a countable intersection of dense open sets, and since the countable intersection of countable intersections is again a countable intersection, by the Baire Category Theorem the intersection  $\bigcap_{i \in \mathbb{N}} G_i$  is a dense  $G_\delta$  subset of  $X$ .  $\square$

**Proposition 5.8.** *Suppose  $X$  is a separable metric space. Then  $X$  can be embedded in the Hilbert cube  $I_\omega$ . Moreover the set of embeddings of  $X$  in  $I_\omega$  contain a dense\*  $G_\delta$  set in the function space  $(C(X, I_\omega), \|\cdot\|_\infty)$ .*

Before we prove Proposition 5.8, we shall first define some new concepts and prove some auxiliary results, of which we will make use.

**Definition 5.9** ( $\mathcal{A}$ -map). Suppose  $X$  and  $Y$  are topological spaces, and that  $\mathcal{A}$  is a finite open covering of  $X$ . We say that the continuous function  $g: X \rightarrow Y$  is an  $\mathcal{A}$ -map, if every point of  $Y$  has a neighbourhood in  $Y$  such that the preimage of this neighbourhood under  $g$  is entirely contained in some member of  $\mathcal{A}$ .

**Definition 5.10** (Basic sequence of coverings). Let  $\mathcal{A}$  be a finite open covering of the topological space  $X$ . Denote by  $S_{\mathcal{A}}(x)$  the open set which is the union of all the members of  $\mathcal{A}$  that contain the point  $x \in X$ . A countable

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\*In other words, making arbitrarily small modifications to a continuous function suffice to make it an embedding.

collection  $\mathcal{A}_1, \mathcal{A}_2, \dots$  of finite open coverings is called a *basic sequence of coverings*, if given a point  $x \in X$  and a neighbourhood  $U$  of  $x$ , at least one of the open sets  $S_{\mathcal{A}_1}(x), S_{\mathcal{A}_2}(x), \dots$  is contained in  $U$ .

**Proposition 5.11.** *For every separable metric space  $X$  there exists a basic sequence of coverings.*

*Proof.* Let  $\{U_i\}_{i=1}^\infty$  be any countable basis of  $X$ . We consider pairs  $U_m, U_n$  of members of this basis for which it holds that

$$\overline{U_n} \subset U_m.$$

Denote by  $\mathcal{A}_{n,m}$  the covering of  $X$  that consists of two members, namely the sets  $X \setminus \overline{U_n}$  and  $U_m$ . The collection consisting of these coverings  $\mathcal{A}_{n,m}$  is countable. Moreover,  $x \in U_n$  implies  $S_{\mathcal{A}_{n,m}}(x) = U_m$ . Thus the collection of sets  $\{S_{\mathcal{A}_{n,m}}(x)\}$  for a given point  $x$  includes the collection of all the  $U_m$  containing  $x$ , which proves that  $\{\mathcal{A}_{n,m}\}$  is a basic sequence of coverings.  $\square$

**Proposition 5.12.** *Suppose  $X$  and  $Y$  are metric spaces, and that  $\mathcal{A}_1, \mathcal{A}_2, \dots$  is a basic sequence of coverings of  $X$ . If the function  $g: X \rightarrow Y$  is an  $\mathcal{A}_i$ -map for every  $i \in \mathbb{N}$ , then  $g$  is an embedding.*

*Proof.* We shall show, that if  $x$  is any point of  $X$  and  $U$  a neighbourhood of  $x$ , there exists a neighbourhood  $V$  of  $g(x)$  such that the preimage of  $V$  under  $g$  is contained in  $U$ . From this follows the injectivity of  $g$  and the continuity of the inverse  $g_1^{-1}: gX \rightarrow X$ , where  $g_1: X \rightarrow gX$  is the function defined by  $g$ .

By the definition of a basic sequence of coverings, there exists an  $\mathcal{A}_i$  for which

$$S_{\mathcal{A}_i}(x) \subset U.$$

Since  $g$  is an  $\mathcal{A}_i$ -map there exists a neighbourhood  $V$  of  $g(x)$  and a set  $U_0^i \in \mathcal{A}_i$  for which

$$g^{-1}V \subset U_0^i.$$

Since  $x \in g^{-1}V \subset U_0^i$ , we have

$$U_0^i \subset S_{\mathcal{A}_i}(x).$$

Thus  $g^{-1}V \subset U$ , which proves the claim.  $\square$

**Proposition 5.13.** *Let  $X$  be a separable metric space and  $Y$  a compact metric space. For each finite open covering  $\mathcal{A}$  of  $X$ , the set  $G_{\mathcal{A}}$  of all  $\mathcal{A}$ -maps from  $X$  to  $Y$  is open in the function space  $(C(X, Y), \|\cdot\|_\infty)$ .*

*Proof.* Suppose  $g: X \rightarrow Y$  is an  $\mathcal{A}$ -map. Hence every point of  $Y$  has a neighbourhood whose preimage under  $g$  is contained in some member of  $\mathcal{A}$ . Since  $Y$  is compact, there is a finite subcollection of these neighbourhoods which form a covering  $\mathcal{C}$  of  $Y$ . From the Lebesgue Covering Theorem (Proposition 2.33) one derives a number  $\lambda > 0$  with the property that any set in  $Y$  whose diameter is less than  $\lambda$  is contained in some member of  $\mathcal{C}$ , and thus has its preimage under  $g$  entirely contained in some member of  $\mathcal{A}$ . Let  $f: X \rightarrow Y$  be a continuous function satisfying

$$\|f - g\|_\infty < \frac{1}{3}\lambda.$$

Take a spherical neighbourhood of diameter  $\frac{1}{3}\lambda$  around every point of  $Y$ . Denote by  $V$  the preimage of one of these spherical neighbourhoods under  $f$ , that is  $V := f^{-1}\left(B(y, \frac{1}{3}\lambda)\right)$  for some  $y \in Y$ . The diameter of the set  $gV$  is less than  $\lambda$ , which is a result of the inequality above. Thus the set  $V$  is contained in some member of  $\mathcal{A}$ , which in turn implies that  $f$  is an  $\mathcal{A}$ -map. This completes the proof.  $\square$

*Proof of Proposition 5.8.* Consider the function space  $(C(X, I_\omega), \|\cdot\|_\infty)$ . Let  $\mathcal{A}_1, \mathcal{A}_2, \dots$  be a basic sequence of coverings of  $X$ , which exists by Proposition 5.11, let  $G_{\mathcal{A}_i}$  be the set of  $\mathcal{A}_i$ -maps of  $X$  into  $I_\omega$  and let

$$H := \bigcap_{i=1}^{\infty} G_{\mathcal{A}_i}.$$

By Proposition 5.12 each element of  $H$  is an embedding. By Proposition 5.13 each  $G_{\mathcal{A}_i}$  is open in  $C(X, I_\omega)$  and thus  $H$  is a  $G_\delta$  set. Thus by Corollary 5.7 we only need to show the following:

*For each finite open covering  $\mathcal{A}$  of  $X$  denote by  $G_{\mathcal{A}}$  the set of  $\mathcal{A}$ -maps from  $X$  to  $I_\omega$ . Then  $G_{\mathcal{A}}$  is dense in  $C(X, I_\omega)$ .*

Suppose  $f \in C(X, I_\omega)$  and let  $\varepsilon > 0$ . We shall construct a continuous function  $g: X \rightarrow I_\omega$  such that

$$\|f - g\|_\infty < \varepsilon \tag{3}$$

$$g \in G_{\mathcal{A}}. \tag{4}$$

As a compact space,  $I_\omega$  has a finite open covering  $\mathcal{K}$  of mesh less than  $\frac{1}{2}\varepsilon$ . Let  $\{U_i\}_{i=1}^r$  be the covering of  $X$  made up of the non-empty sets of the form

$$A \cap f^{-1}K,$$

where  $A \in \mathcal{A}$  and  $K \in \mathcal{K}$ . Thus for every  $U_i$  it holds that  $d(fU_i) < \frac{1}{2}\varepsilon$ .

We shall select linearly independent points  $p_1, p_2, \dots, p_r \in I_\omega$  for which it holds that

$$d(p_i, fU_i) < \frac{1}{2}\varepsilon, \quad i = 1, 2, \dots, r. \quad (5)$$

For each  $x \in X$  set  $w_i(x) = d(x, X \setminus U_i)$ , where  $i \in \{1, 2, \dots, r\}$  with the understanding, that  $w_i(x) = 1$ , if  $U_i = X$ . Now  $w_i(x) > 0$  if  $x \in U_i$  and  $w_i(x) = 0$  if  $x \notin U_i$ . For every  $x \in X$  at least one  $w_i(x)$  is positive, since  $\{U_i\}_{i=1}^r$  is a covering of  $X$ . Thus the function  $g: X \rightarrow I_\omega$  defined with the rule

$$g(x) = \frac{1}{\sum_{i=1}^r w_i(x)} \sum_{i=1}^r w_i(x) p_i$$

is well defined, and evidently continuous.

We shall next show, that  $g$  satisfies (3) and (4). Let  $x \in X$  and suppose that the collection  $\{U_i\}_{i=1}^r$  is so numbered, that  $U_1, U_2, \dots, U_s$  are the sets that contain the point  $x$ . Then  $w_i(x) > 0$  for  $i \leq s$ , and  $w_i(x) = 0$  for  $i > s$ . Hence when examining the point  $g(x)$ , we only need to consider the points  $p_1, p_2, \dots, p_s$ . Since  $x \in U_i$ , when  $i \leq s$ , and from the fact that  $d(fU_i) < \frac{1}{2}\varepsilon$ , together with (5) we get

$$d(p_i, f(x)) < \varepsilon, \quad i \leq s.$$

Thus the point  $g(x)$  satisfies

$$d(g(x), f(x)) < \varepsilon.$$

Suppose  $U_{i_1}, U_{i_2}, \dots, U_{i_k}$  are all of the members of  $\{U_i\}_{i=1}^r$  that contain a given point  $x \in X$ . Consider the affine subspace  $A(x)$  of the Hilbert cube spanned by the points  $p_{i_1}, p_{i_2}, \dots, p_{i_k}$ . In other words,  $A(x)$  consists of exactly those points of the Hilbert cube, that can be represented in the form

$$\sum_{j=1}^k \lambda_{i_j} p_{i_j},$$

where  $\sum_{j=1}^k \lambda_{i_j} = 1$ .

Clearly the point  $g(x)$  is contained in  $A(x)$ . Let  $x'$  be another point of the space  $X$ . We claim, that if  $A(x) \cap A(x') \neq \emptyset$ , these affine subspaces contain some common point  $p_i$ , and thus  $x$  and  $x'$  are contained in a common member of  $\{U_i\}_{i=1}^r$ . Otherwise, from the equation

$$\sum_{j=1}^k \alpha_{i_j} p_{i_j} = \sum_{j=1}^m \beta_{l_j} p_{l_j},$$

where  $\sum_{j=1}^k \alpha_{i_j} = \sum_{j=1}^m \beta_{l_j} = 1$ , it would follow that

$$\sum_{j=1}^k \alpha_{i_j} p_{i_j} - \sum_{j=1}^m \beta_{l_j} p_{l_j} = \bar{0}$$

and thus the points  $p_1, p_2, \dots, p_r$  we chose before would not be linearly independent, a contradiction.

Since there are only a finite number of these affine subspaces  $A(x)$ , there exists a number  $\delta > 0$  such that any two of these affine subspaces  $A(x)$  and  $A(x')$  either meet, or have a distance  $\geq \delta$  from each other. If

$$d(g(x), g(x')) < \delta,$$

it certainly holds that  $d(A(x), A(x')) < \delta$ , and as was noted above, this shows that  $x$  and  $x'$  are contained in a common member of  $\{U_i\}_{i=1}^r$ . This shows, that  $g$  is an  $\mathcal{A}$ -map, and thus the set  $G_{\mathcal{A}}$  is dense in  $C(X, I_{\omega})$ .  $\square$

**Proposition 5.14.** *Suppose  $X$  is a separable metric space with  $\text{ind}(X) \geq n$ , where  $n \geq 1$ . Then there exists a continuous function  $f: X \rightarrow I^n$  such that  $f$  has at least one stable value.*

*Proof.* Suppose that no continuous function  $f: X \rightarrow I^n$  has stable values. Now, by the definition of unstable values, for each point  $y \in fX$ , the function  $f$  can be approximated arbitrarily closely by a continuous function  $g': X \rightarrow I^n$ , for which  $y \notin g'X$ . Proposition 5.4 gives us in turn an arbitrarily close approximation of  $g'$ , namely a function  $g: X \rightarrow I^n$ , with the property  $y \notin \overline{gX}$ .

Let us consider the function space  $C(X, I_{\omega})$  with the metric induced by the sup-norm. Let  $M = M(i_1, i_2, \dots, i_n; c_1, c_2, \dots, c_n)$  be the subspace of the Hilbert cube defined by the  $n$  equations

$$x_{i_1} = c_1, x_{i_2} = c_2, \dots, x_{i_n} = c_n. \quad (6)$$

We shall denote by  $G_M$  the subset of  $C(X, I_{\omega})$ , consisting of functions  $g$  with the property

$$\overline{gX} \subset I_{\omega} \setminus M.$$

The set of functions  $G_M$  is dense in  $C(X, I_{\omega})$ . To show this, let  $f \in C(X, I_{\omega})$ . Now we have

$$f(x) = (f_1(x), f_2(x), \dots), \quad |f_i(x)| \leq 1/i, \quad \text{where } i \in \mathbb{N}. \quad (7)$$

The functions  $f_{i_1}, f_{i_2}, \dots, f_{i_n}$  define a continuous function  $\bar{f}: X \rightarrow I^n$ , and as we remarked above, there exists an arbitrarily close approximation  $f'$  of  $\bar{f}$ , such that  $(c_1, c_2, \dots, c_n) \notin \overline{f'X}$ . Thus,  $G_M$  is dense in  $C(X, I_{\omega})$ .



To show that  $G_M$  is open in  $C(X, I_\omega)$  consider a function  $g \in G_M$ . Now the distance  $d := d(g(X), M)$  is positive, and thus every function  $f \in C(X, I_\omega)$  with  $\|f - g\|_\infty < d$  belongs to  $G_M$ .

Let us now focus on the functions  $g \in C(X, I_\omega)$  with the property

$$\overline{gX} \subset \mathcal{M}_\omega^{n-1}, \quad (8)$$

where  $\mathcal{M}_\omega^{n-1}$  denotes the points in the Hilbert cube at most  $n - 1$  whose coordinates are rational (see Example 3.21). The complement of  $\mathcal{M}_\omega^{n-1}$  is the countable union of hyperplanes  $M_1, M_2, \dots$  of type (6), namely those corresponding to all possible combinations of  $n$  indexes  $i_j$  and rational numbers  $c_j$ . Thus (8) is equivalent to

$$g \in G_{M_i}, \quad \text{for every } i \in \mathbb{N}.$$

By Proposition 5.8 the space  $C(X, I_\omega)$  contains a dense  $G_\delta$  set  $E$ , each member of which is an embedding. The set

$$E' := E \cap \left( \bigcap_{i \in \mathbb{N}} G_{M_i} \right)$$

is dense in  $C(X, I_\omega)$  as the countable intersection of dense  $G_\delta$  sets by Corollary 5.7. In particular,  $E'$  is non-empty.

Thus there exists an embedding  $h$ , which embeds  $X$  in  $\mathcal{M}_\omega^{n-1}$ . Hence by Theorem 3.15 and Proposition 3.16 we have  $\text{ind}(X) \leq \text{ind}(\mathcal{M}_\omega^{n-1})$ . In Example 3.21 we in fact showed that  $\text{ind}(\mathcal{M}_\omega^{n-1}) \leq n - 1$ , thus we have contradicted the assumption that  $\text{ind}(X) \geq n$ .  $\square$

**Proposition 5.15.** *Suppose  $X$  is a metric space and let  $f: X \rightarrow I^n$  be continuous. An interior point  $y$  of  $fX$  is an unstable value of  $f$ , if and only if for every neighbourhood  $U$  of  $y$ , there exists a continuous map  $g: X \rightarrow I^n$  satisfying*

$$g(x) = f(x) \quad \text{if } f(x) \notin U \quad (9)$$

$$g(x) \in U \quad \text{if } f(x) \in U \quad (10)$$

$$y \notin gX. \quad (11)$$

*Proof.* Suppose that the condition of the proposition holds. From (9) and (10) it follows that for every neighbourhood  $U$  of the point  $y$  there exists a continuous function  $g: X \rightarrow I^n$  such that

$$\|f(x) - g(x)\| \leq d(U)$$

for every  $x \in X$ , and  $gX \subset I^n \setminus \{y\}$ . Thus  $y$  is an unstable value of  $f$ .

Suppose then that  $y$  is an interior point of  $I^n$  and that  $y$  is an unstable value of the function  $f$ . Let  $\varepsilon > 0$ . Put  $U := B(y, \varepsilon)$ . Since  $y$  is an unstable value of  $f$ , there exists a continuous function  $g': X \rightarrow I^n$  such that

$$\|f(x) - g'(x)\| < \varepsilon/2 \quad (12)$$

$$g'(x) \neq y \quad (13)$$

for all  $x \in X$ . We shall construct a new function  $g$  as follows:

$$g(x) = g'(x) \quad \text{if } \|f(x) - y\| \leq \varepsilon/2, \quad (14)$$

$$g(x) = 2 \left( 1 - \frac{\|f(x) - y\|}{\varepsilon} \right) g'(x) - \left( 1 - \frac{2\|f(x) - y\|}{\varepsilon} \right) f(x) \quad (15)$$

if  $\varepsilon/2 \leq \|f(x) - y\| \leq \varepsilon$ , and

$$g(x) = f(x) \quad \text{if } \|f(x) - y\| \geq \varepsilon. \quad (16)$$

The subfunctions that define  $g$  are all continuous and their domains are closed in  $X$ , since the domains are preimages of closed sets under the continuous function  $f$ . If in the intersections of the domains of these subfunctions, the subfunctions receive the same values, the piecewise defined function  $g$  is continuous. From the definitions of the subfunctions, it is evident that we only need to check the cases  $\|f(x) - y\| = \varepsilon/2$  and  $\|f(x) - y\| = \varepsilon$ .

In the first case  $g(x) = g'(x)$  and

$$g(x) = 2 \left( 1 - \frac{\varepsilon/2}{\varepsilon} \right) g'(x) - \left( 1 - \frac{2\varepsilon/2}{\varepsilon} \right) f(x) = g'(x).$$

In the second case we have

$$g(x) = 2 \left( 1 - \frac{\varepsilon}{\varepsilon} \right) g'(x) - \left( 1 - \frac{2\varepsilon}{\varepsilon} \right) f(x) = f(x),$$

and  $g(x) = f(x)$ . We have thus established, that indeed  $g: X \rightarrow I^n$  is a continuous function.

Condition (9) is the same as (16). If  $\varepsilon/2 \leq \|f(x) - y\| \leq \varepsilon$  we get from (12) and (15) that

$$\begin{aligned} \|g(x) - y\| - \|f(x) - y\| &\leq \|f(x) - g(x)\| \\ &= \left\| f(x) - g'(x) + (f(x) - g'(x)) \left( 1 - \frac{2\|f(x) - y\|}{\varepsilon} \right) \right\| \\ &= 2 \left( 1 - \frac{\|f(x) - y\|}{\varepsilon} \right) \|f(x) - g'(x)\| \\ &< \varepsilon - \|f(x) - y\|, \end{aligned}$$

and thus

$$0 < \|g(x) - y\| < \varepsilon. \quad (17)$$

By (12), (13) and (14) the inequality (17) also holds when  $\|f(x)\| \leq \varepsilon/2$ . Hence the conditions (10) and (11) hold, completing the proof.  $\square$

**Definition 5.16** (Extendable function). If for a continuous function  $f: M \rightarrow Y$  defined on a subspace  $M$  of a space  $X$ , there exists a continuous function  $F: X \rightarrow Y$ , such that  $F(x) = f(x)$  for all  $x \in M$ , we say that  $f$  is *continuously extendable*, or more briefly *extendable*, and call  $F$  an *extension* of  $f$  over  $X$ .

We have already encountered a result concerning extensions, namely Urysohn's lemma (Proposition 2.18). It states that if a subspace  $M$  of a normal space  $X$  can be represented as a union of two disjoint closed sets  $A, B \subset X$ , the function  $f: M \rightarrow [0, 1]$ , defined by  $f(x) = 0$ , when  $x \in A$  and  $f(x) = 1$  when  $x \in B$  is continuously extendable over  $X$ . Actually a much more general theorem holds, which is the Tietze Extension Theorem. Before we prove this theorem, we first state and prove a small lemma.

**Lemma 5.17.** *Suppose  $X$  is a normal space and  $A \subset X$  is closed,  $a > 0$  and that the function  $f: A \rightarrow [-a, a]$  is continuous. Now there exists a continuous function  $h: X \rightarrow [-a/3, a/3]$  such that  $|f(x) - h(x)| \leq 2a/3$  for all  $x \in A$ .*

*Proof.* The sets  $A_1 := f^{-1}[-a, -a/3]$  and  $A_2 := f^{-1}[a/3, a]$  are closed and disjoint. By Urysohn's lemma (Proposition 2.18), there exists a continuous function  $h: X \rightarrow [-a/3, a/3]$  such that  $hA_1 \subset \{-a/3\}$  and  $hA_2 \subset \{a/3\}$ . Now  $h$  is the desired function.  $\square$

**Theorem 5.18** (Tietze Extension Theorem). *Every continuous function from a closed subspace  $M$  of a normal space  $X$  to a closed interval  $[a, b]$  is continuously extendable over  $X$ .*

*Proof.* Let  $f: M \rightarrow [a, b]$  be continuous. Since the closed interval  $[a, b]$  is homeomorphic to  $[-1, 1]$  we may assume  $[a, b] = [-1, 1]$  for simplicity. We shall apply Lemma 5.17 to define a sequence  $g_1, g_2, \dots$  of continuous functions from  $X$  to  $[-1, 1]$  such that

$$|g_i(x)| \leq \frac{1}{3} \left(\frac{2}{3}\right)^{i-1}, \quad \text{for } x \in X \quad (18)$$

and

$$|f(x) - \sum_{j=1}^i g_j(x)| \leq \left(\frac{2}{3}\right)^i, \quad \text{for } x \in M. \quad (19)$$

To obtain  $g_1$ , we apply Lemma 5.17 to the function  $f$ . Thus

$$|g_1(x)| \leq \frac{1}{3}, \quad \text{for } x \in X$$

and

$$|f(x) - g_1(x)| \leq \frac{2}{3}, \quad \text{for } x \in M.$$

Suppose we have defined the functions  $g_1, g_2, \dots, g_i$ . Applying Lemma 5.17 to the function  $f - \left(\sum_{j=1}^i g_j\right)|_M$  we obtain a function  $g_{i+1}$  satisfying

$$|g_{i+1}(x)| \leq \frac{1}{3} \left(\frac{2}{3}\right)^i, \quad \text{for } x \in X$$

and

$$\left|f(x) - \sum_{j=1}^{i+1} g_j(x)\right| \leq \left(\frac{2}{3}\right)^{i+1}, \quad \text{for } x \in M.$$

From (18) and the Weierstraß test (see Väisälä [7, Lause 10.14, p. 81]) it follows that the formula  $F(x) := \sum_{i=1}^{\infty} g_i(x)$  defines a continuous function  $F: X \rightarrow [-1, 1]$  (see Väisälä [7, Lause 10.13, p. 81]), and (19) implies  $F(x) = f(x)$  for all  $x \in M$ , so that  $F$  is an extension of  $f$  over  $X$ .  $\square$

**Corollary 5.19.** *Every continuous function from a closed subspace  $M$  of a normal space  $X$  to the cube  $I^n$  is continuously extendable over  $X$ .*

*Proof.* This follows from applying the Tietze Extension Theorem to each of the coordinate functions.  $\square$

**Corollary 5.20.** *Suppose  $M$  is a closed subset of a normal space  $X$ , and let  $f: M \rightarrow S^n$  be a continuous function. Now there exists a neighbourhood of the set  $M$  over which  $f$  can be extended.*

*Proof.* Since  $S^n$  has radius 1, we have for the coordinates of the point  $f(x) \in \mathbb{R}^{n+1}$

$$\sum_{i=1}^{n+1} f_i(x)^2 = 1,$$

and thus

$$|f_i(x)| \leq 1 \quad \text{for } i = 1, 2, \dots, n+1.$$

Hence we can apply the Tietze Extension Theorem (Theorem 5.18) to each of the coordinate functions  $f_i: M \rightarrow [-1, 1]$ . Thus for each  $f_i$  we get extensions  $F_i: X \rightarrow [-1, 1]$ .

Let  $U \subset X$  denote the set of points for which

$$\sum_{i=1}^{n+1} F_i(x)^2 > 0.$$

Thus

$$U = \bigcup_{i=1}^{n+1} F_i^{-1}([-1, 1] \setminus \{0\}),$$

which makes  $U$  an open subset of  $X$  as a union of open sets. It is evident that  $M \subset U$ . We shall define the coordinate functions of the function  $G: U \rightarrow S^n$  with the rule

$$G_i(x) = \frac{F_i(x)}{\left(\sum_{i=1}^{n+1} F_i(x)^2\right)^{1/2}} \quad \text{for } i = 1, 2, \dots, n+1.$$

The mapping  $G$  is the desired extension of  $f$  over  $U$ . □

**Proposition 5.21.** *A separable metric space  $X$  has  $\text{ind}(X) \leq n$ , if and only if for each closed set  $C \subset X$  and continuous function  $f: C \rightarrow S^n$  there exists an extension of  $f$  over  $X$ .*

*Proof.* Suppose  $f$  is a continuous function from a closed subset  $C \subset X$  to  $S^n$ . Since  $S^n$  and  $\partial I^{n+1}$  are homeomorphic, we think of  $f$  as function from  $C$  to  $I^{n+1}$ . By Corollary 5.19 there exists a continuous function  $F': X \rightarrow I^{n+1}$ , which is an extension of  $f$ .

Suppose that  $\text{ind}(X) \leq n$ . Proposition 5.3 implies, that the origin is not a stable value of  $F'$ . Proposition 5.15 gives us a continuous function  $F'': X \rightarrow I^{n+1}$  such that  $\bar{0} \notin F''X$ , while  $F''(x) = F'(x)$  for all  $x \in X$  that do not map to the interior of  $I^{n+1}$  under  $F'$ . In particular, for  $x \in C$ , we have

$$F''(x) = F'(x) = f(x).$$

Let  $F: X \rightarrow \partial I^{n+1}$  be the projection of the point  $F''(x)$  from the origin on the boundary  $\partial I^{n+1}$ . Now  $F$  is the desired extension of  $f$ .

Suppose the condition of the proposition holds. Now, in order to prove that  $\text{ind}(X) \leq n$ , it is enough to show, according to Proposition 5.14, that a continuous function  $f: X \rightarrow I^{n+1}$  cannot have stable values. Example 5.2 tells us that a boundary point of  $I^{n+1}$  is never stable. Hence it suffices to show that the interior points of  $I^{n+1}$  cannot be stable. Let  $y$  be an interior point of  $I^{n+1}$  and let  $0 < \varepsilon < 1$ . Denote by  $C$  the inverse image  $f^{-1}\left(\partial B(y, \frac{\varepsilon}{2})\right)$ . As the function  $f$  is continuous,  $C$  is a closed subset of  $X$ . Let  $\bar{f}$  denote the restriction of  $f$  to  $C$ . By hypothesis, there exists a continuous function  $F: X \rightarrow \partial B(y, \frac{\varepsilon}{2})$  such that  $F(x) = \bar{f}(x)$  for  $x \in C$ .

We shall construct a new function  $g$  defined in the whole space  $X$ , taking values in  $I^{n+1}$  with the rules

$$\begin{aligned} g(x) &= f(x) && \text{if } f(x) \notin B\left(y, \frac{\varepsilon}{2}\right), \\ g(x) &= F(x) && \text{if } f(x) \in \overline{B}\left(y, \frac{\varepsilon}{2}\right). \end{aligned}$$

In other words,  $g(x) = f(x)$ , whenever  $x \in f^{-1}\left(I^{n+1} \setminus B\left(y, \frac{\varepsilon}{2}\right)\right)$  and  $g(x) = F(x)$ , when  $x \in f^{-1}\left(\overline{B}\left(y, \frac{\varepsilon}{2}\right)\right)$ . The intersection of these two pre-images is the set  $C$  defined above. By the definition of the function  $F$ , we have  $f(x) = F(x)$ , if  $x \in C$ , and thus the function  $g$  is continuous.

Now  $g: X \rightarrow I^{n+1} \setminus B\left(y, \frac{\varepsilon}{2}\right)$  is a continuous function with  $\|f - g\|_\infty < \varepsilon$ . This completes the proof.  $\square$

**Corollary 5.22.** *Suppose  $C$  is a closed subset of the separable metric space  $X$ . If  $\text{ind}(X \setminus C) \leq n$ , then every continuous function  $f: C \rightarrow S^n$  can be extended over  $X$ .*

*Proof.* Suppose  $f: C \rightarrow S^n$  is continuous. Corollary 5.20 shows that there exist an open set  $U \supset C$  and an extension  $f'$  of  $f$  over  $U$ . Since  $X$  is normal, by Proposition 2.16, there exists an open set  $V \subset X$  satisfying

$$C \subset V \subset \overline{V} \subset U.$$

Let us consider the restriction

$$\bar{f} := f'|_{\overline{V} \cap (X \setminus C)}.$$

This is a continuous function from a closed subset of the space  $X \setminus C$  to the sphere  $S^n$ . Now the subspace  $X \setminus C$  has dimension  $\leq n$ , and by Proposition 5.21 there exists an extension  $f''$  of  $\bar{f}$  over  $X \setminus C$ . By putting

$$\begin{aligned} F(x) &= f(x) && \text{if } x \in C, \\ F(x) &= f''(x) && \text{if } x \in X \setminus C, \end{aligned}$$

we get the desired extension  $F: X \rightarrow S^n$  of  $f$ .  $\square$

Let  $A$  be a subset of an arbitrary topological space  $X$ . If  $f: X \rightarrow X$  is a homeomorphism, then all interior points of  $A$  map to the interior points of  $fA$ , and conversely, if  $f(x)$  is an interior point of  $fA$ , then  $x \in \text{int}A$ . However, if we consider an embedding  $h: A \rightarrow X$ , it is not generally true, that  $h$  maps the interior points of  $A$  to the interior of  $hA$ . If the space  $X$  is a Euclidean space we in fact have

**Theorem 5.23** (Brouwer's Theorem on the Invariance of Domain). *Suppose  $A$  is a subset of the Euclidean space  $\mathbb{R}^n$  and let  $h: A \rightarrow \mathbb{R}^n$  be an embedding. Then, if  $x \in A$  is an interior point of  $A$ , the point  $h(x)$  belongs to the interior of  $hA$ . In particular, if  $U$  and  $V$  are homeomorphic subsets of  $\mathbb{R}^n$  and  $U$  is open, then  $V$  is open.*

We shall prove Brouwer's Theorem on the Invariance of Domain with the help of the proposition below:

**Proposition 5.24.** *Let  $A \subset \mathbb{R}^n$  be compact. The point  $x$  is a boundary point of  $A$ , if and only if the point  $x$  has arbitrarily small neighbourhoods  $U \subset A$  open in  $A$ , such that any continuous function  $f: A \setminus U \rightarrow S^{n-1}$  can be extended over  $A$ .*

*Proof.* Suppose  $x \in \partial A$ . Let  $\varepsilon > 0$ , and put  $B(x) := B^n(x, \varepsilon)$  and  $U := A \cap B(x)$ . We will show that the set  $U$  has the desired property. Because the set  $A \setminus U = A \cap \mathbb{C}U$  is compact, it is closed in any containing space. Thus by Corollary 5.22 any continuous function  $f: A \setminus U \rightarrow \partial B(x)$  can be extended to a function  $f': (A \setminus U) \cup \partial B(x) \rightarrow \partial B(x)$ . Let  $q$  be a point of  $B(x)$  that is not in  $A$ . For each  $y \in A$  denote by  $y'$  the projection of  $y$  from the point  $q$  to the boundary  $\partial B(x)$ . We define

$$\begin{aligned} F(y) &= f'(y') & \text{if } y \in U, \\ F(y) &= f(y) & \text{if } y \in A \setminus U. \end{aligned}$$

The function  $F: A \rightarrow \partial B(x)$  is the desired extension.

Suppose  $x$  is an interior point of  $A$ . Let  $\varepsilon > 0$  be such that  $\overline{B^n}(x, \varepsilon) \subset A$ . We again denote  $B^n(x, \varepsilon)$  by  $B(x)$ . We shall show that for any neighbourhood  $U \subset B(x)$  of  $x$ , there exists a continuous function  $f: A \setminus U \rightarrow \partial B(x)$ , that cannot be extended over  $A$ . Let  $f$  be the projection of  $A \setminus U$  on the boundary  $\partial B(x)$  from the point  $x$ . Suppose we can extend  $f$  over  $A$ . Let us denote this extension by  $\bar{f}$ . Now the restriction  $\bar{f}|_{\overline{B(x)}}$  is a continuous function from a closed ball to the sphere  $\partial B(x)$ , which leaves each of the boundary points fixed, thus contradicting Corollary 4.17. This completes the proof.  $\square$

*Proof of Theorem 5.23.* We may assume that  $A \subset \mathbb{R}^n$  is compact, since an interior point  $x$  of  $A$  is also an interior point of some ball  $\overline{B^n}(x, \varepsilon)$ , where  $\varepsilon > 0$ . Since the image of a compact set under a continuous function is compact by Väisälä [6, Lausé 13.18, p. 100], we can apply Proposition 5.24 to the boundary points of  $hA$ .

Suppose, that  $x \in \text{int}A$ , and suppose, for a contradiction, that  $h(x) \in \partial(hA)$ . We denote by  $h_1: A \rightarrow hA$  the homeomorphism defined by the

embedding  $h$ . Let  $U \subset hA$  be some neighbourhood of  $h(x)$  in  $hA$ . Now the inverse image  $h^{-1}U$  is a neighbourhood of the point  $x$  in  $A$ , and by the normality of  $A$ , there exists a number  $\varepsilon > 0$  such that  $\overline{B^n}(x, \varepsilon) =: \overline{B} \subset h^{-1}U$ .

Let  $f: hA \setminus U \rightarrow \partial \overline{B}$  be a function which maps every  $y \in hA \setminus U$  to the boundary  $\partial \overline{B}$  in such a way, that the point  $f(y)$  corresponds to the projection of  $h_1^{-1}(y)$  to the boundary  $\partial \overline{B}$  from the point  $x$ . Clearly,  $f$  is continuous as a composition of two continuous functions.

We may apply Proposition 5.24. Hence there exists an extension of  $f$ , namely some function  $\bar{f}: hA \rightarrow \partial \overline{B}$ . Now the composition  $\bar{f} \circ (h|_{\overline{B}})$  is a continuous function from a closed ball to its boundary, which keeps each boundary point fixed thus contradicting Corollary 4.17.

We have shown that the embedding  $h$  maps each interior point of  $A$  to the interior of  $hA$ , which completes the proof of Brouwer's Theorem on the Invariance of Domain.  $\square$



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